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AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

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THE APRIL MEETING OF THE SAN FRANCISCO SECTION

The forty-first regular meeting of the San Francisco Section of the American Mathematical Society was held at Stanford University on April 7. Professor Cajori presided. The total attendance was twenty-three, including the following seventeen members of the Society:

Alderton, Allardice, Bernstein, Blichfeldt, Buck, Cajori, A. F. Carpenter, Growe, Hoskins, Lehmer, Levy, Libby, McEwen, F. R. Morris, Smail, Pauline Sperry, A. R. Williams.

The Secretary read a communication from Secretary Richardson asking the members of the Society to cooperate with the Endowment Committee, headed by Professor Coolidge, in the forthcoming campaign for funds for the Society.

In response to a request by a committee from the University of Washington for more active participation in the work of the Section by members of the Society living in and about Seattle, it was decided to hold a special meeting of the Section at Seattle in December, 1923, with Professor Carpenter acting as Secretary.

It was decided to hold the next regular meeting of the Section at Los Angeles, September 17–20, in connection with the meeting of the Pacific Division of the American Association for the Advancement of Science.

Titles and abstracts of papers read at this meeting follow. The papers of Professors Bell and Neikirk were read by title.

1. Professor E. T. Bell: *Euler algebra.*

This name is given to the algebra of certain infinite processes fundamental in the study of the multiplicative properties of numbers. Two other algebras, called here the Cauchy and Dirichlet varieties, are also investigated. The first is that of power series, the second that of Dirichlet series. Euler algebra is a species of resultant between these two. The paper is to appear in the TRANSACTIONS OF THIS SOCIETY.

2. Professor E. T. Bell: *A class of numbers connected with partitions.*

The theory of partitions and that of the representation of an integer as a sum of squares originating in the elliptic modular functions must be closely interconnected. The connections depend upon eight new systems of integers. These integers are functions of two parameters. For values $\geqslant 6$ of the parameters the numbers are connected with the class numbers of binary quadratic forms of a negative determinant. The paper will be published in the AMERICAN JOURNAL.

3. Professor E. T. Bell: *Square-partition congruences.*

This paper will appear in full in an early issue of this BULLETIN.

4. Professor E. T. Bell: *Theta functions and arithmetic.*

Any relation whatever between theta functions of $p > 1$ arguments implies and is implied by a corresponding relation between the solutions of a set of $\frac{1}{2}p(p+1)$ indeterminate equations of the following sort: each equation in the set is restricted only in that its coefficients must be (arbitrary) integers; the equations are not necessarily homogeneous; the degrees of the several equations in the set may be any $\frac{1}{2}p(p+1)$ positive integers, and likewise for the numbers of indeterminates. The excepted case $p = 1$ is treated in a forthcoming paper in the TRANSACTIONS OF THIS SOCIETY.

5. Professor E. T. Bell: *Note on total representations as sums of squares.*

If $O(r)$, $E(r)$ denote respectively the total numbers of representations of r as the sum of an odd, an even number of squares with roots different from zero, $O(2n+1) > E(2n+1)$, $E(2n) > O(2n)$ for all positive integers n.

6. Professor B. A. Bernstein: *Complete set of representations of two-element algebras.*

The author gives simple arithmetic representations of all two-element abstract algebras, with some applications to fundamental problems connected with postulate-sets.

7. Professor H. F. Blichfeldt: *Note on quadratic forms.* Preliminary report.

Given a positive quadratic form f of determinant D in n variables. Integers, not all zero, may be substituted for the

variables in this form such that the numerical value of f is $\leqq \gamma_n \mid D \mid^{1/n}$, where γ_n is a function of n only. For $n < 6$, the lowest permissible values of γ_n have been found; γ_6 was proved $< \sqrt{3}$ by Korkine and Zolatareff in MATHEMATISCHE ANNALEN, vol. 6 (1873), p. 378. The present author proves that

$$\gamma_6 < \{8\sqrt[4]{72}\}^{1/6} = 1.690\cdots.$$

8. Professor Florian Cajori: *Varieties of minus signs.*

It is shown that besides the sign $-$ there was introduced, in 1525, $\div$ as a minus sign. This maintained its place in a few mathematical books until the latter part of the nineteenth century. Other minus signs found in books are $\doteqdot, --, ---, \sim$.

9. Professor Florian Cajori: *The evolution of the modern exponential notation.*

The advance toward our modern exponential notation made by the forerunners of Descartes, namely by Chuquet, Bombelli, Digges, Cataldi, Stevin, Romanus, Girard, Bürgi, Reymer, Kepler, Schoner, Hérigone and Hume, is traced with fullness of detail; also the spread of the cartesian symbols and of Newton's extension of it.

10. Professor A. F. Carpenter: *Flecnodal properties of a ruled surface.*

This paper has to do with certain quadrics connected with the flecnode and complex curves of a ruled surface, together with four related space cubics. The theorems deduced involve projective properties of these curves and surfaces, the method of attack being that of the projective differential geometry. The paper will appear in the TÔHOKU JOURNAL.

11. Professor D. N. Lehmer: *On congruences connected with magic squares with odd number of cells.*

The familiar *uniform step* method of constructing odd magic squares is based on certain interesting congruences which seem not to have been studied. One can tell by means of these congruences whether a square constructed with a given *step* and a given *breakstep* will be magic, and whether it also will be *diabolic* and *symmetric.*

12. Professor L. I. Neikirk: *Some non-associative linear algebras.*

The units of these algebras are transformations in a Galois field, and are finite in number. Two kinds of third and higher

powers of units may exist, according to the way the factors are associated, and may be designated as right and left handed powers. Division is not always possible; a unit may not have either a right hand or a left hand inverse.

13. Professor L. L. Smail: *Some theorems on uniform convergence of infinite products.*

Using theorems by Bôcher and Birkhoff on the uniform convergence of $\sum u_n$ and $\sum |u_n|$ (ANNALS OF MATHEMATICS, vol. 4, p. 159, vol. 6, p. 90), a set of theorems are developed giving relations between $\Pi(1 + u_n)$, $\Pi(1 + |u_n|)$, $\sum u_n$, and $\sum |u_n|$.

14. Professor L. L. Smail: *Note on derivatives of a vector product.*

An interesting and simple formula for the nth derivative of the cross- or vector product of two vectors is obtained.

15. Professor L. L. Smail: *Report on a synopsis of the theory of summable infinite processes.*

The author reports that he has in preparation a book on *Synopsis of the Theory of Summable Infinite Processes,* which is to cover exhaustively the field of summable series and other processes, by presenting without proof all the important results, definitions and theorems thus far obtained. This would do for the subject of summability what Dickson's *History of the Theory of Numbers* has done for that subject.

16. Dr. Pauline Sperry: *On real elements in imaginary linear spaces.*

In this paper the author shows that all real points in an imaginary linear k-space lie in a real linear $(k - 1)$-space, also that this real $(k - 1)$-space exists if and only if the imaginary k-space lies in a real $(k + 1)$-space, and obtains the theorems corresponding to these by duality. A number of theorems follow concerning necessary and sufficient conditions for the determination of real spaces by sets of imaginary elements.

B. A. BERNSTEIN,
Secretary of the Section.

THE APRIL MEETING OF THE SOCIETY IN CHICAGO

The nineteenth western meeting of the Society being the fifty-first regular meeting of the Chicago Section, was held at the University of Chicago on Friday and Saturday, April 13 and 14, 1923. There were about eighty persons present at this meeting, among them the following sixty members of the Society:

Ballantine, E. M. Berry, Bliss, Blumberg, Brahana, Brink, C. C. Camp, Coble, Copenhaver, H. B. Curtis, Curtiss, Denton, Dickson, Dresden, Emch, Feldstein, Glover, L. M. Graves, W. L. Hart, M. G. Haseman, Hildebrandt, Hull, Ingraham, Dunham Jackson, Kazarinoff, Kinney, Krathwohl, Lefschetz, Logsdon, N. B. MacLean, MacMillan, Marshall, T. E. Mason, McEwen, J. V. McKelvey, G. A. Miller, Miser, E. H. Moore, E. J. Moulton, F. R. Moulton, C. I. Palmer, Pounder, Rechard, H. L. Rietz, Roman, Roth, Schottenfels, Sinclair, Slaught, Edwin R. Smith, Stetson, R. B. Stone, J. H. Taylor, E. L. Thompson, Townsend, J. S. Turner, Van Vleck, F. E. Wood, Yanney, J. W. A. Young.

The meeting was opened at 10 A.M. on Friday by Professor E. H. Moore, who presided during the first part of the morning session, after which Professor D. R. Curtiss took the chair. During the sessions of Friday afternoon and Saturday morning Professor A. B. Coble, Chairman of the Section, presided. On Friday afternoon Professor S. Lefschetz gave a symposium lecture on *Curves traced on algebraic surfaces*; this lecture will appear in the June number of this BULLETIN. Upon motion of Professor E. H. Moore, the Section expressed to Professor Lefschetz its appreciation of this lecture. The Section voted to hold the Christmas meeting of 1923 in Cincinnati, in affiliation with the meetings of the American Association for the Advancement of Science.

The papers read at this meeting are listed below. The paper of Mr. Davis was read by Professor Dresden. The papers of Blumberg and Barnett were read by title.

1. Professor H. L. Rietz: *On the representation of a certain fundamental law of probability of Laplace.*

This paper first develops the formula of Laplace for the frequency distribution of the sum of n elements each taken at random from an interval 0 to a by a simple geometrical method involving mathematical induction. It next shows that the approximation which Laplace used in his application

of this formula to the problem of the random distribution of the inclination of the orbits of comets is given by the first two non-vanishing terms of the Gram-Charlier representation of an arbitrary continuous frequency function. The paper next develops a more accurate approximation from the Gram-Charlier representation in the sense of a certain least squares criterion. The main difficulty and much of the interest in the paper consists in obtaining remarkably simple expressions for moments of area under our theoretical frequency curve. The auxiliary theorems proved in this connection seem to be of interest in combinatorial analysis.

2. Professor F. E. Wood: *An application of areal coordinates to mixtures and the graphical solution of equations.*

In this paper the author derives the well known results involving the use of points within an equilateral triangle to represent mixtures of three substances; a different method of approach enables him to extend the representation to diluted mixtures of three substances and to general triangles. Some theorems, apparently new, are proved and a graphical method is obtained for finding the percentages of three given mixtures which need to be taken to form a desired mixture of the three fundamental substances. Another application is a method of finding graphically the solution of three simultaneous linear non-homogeneous equations in three unknowns.

3. Dr. H. R. Brahana: *A theorem concerning unit matrices with integral elements.*

Given a matrix M, the normal form of a skew-symmetric matrix of determinant 1 and with integral elements. Poincaré stated without proof that any matrix A with integral elements which satisfies the relation $AMA' = M$ can be considered as a product of matrices of two particular types. In this paper the theorem is proved and it is further shown that any matrix D which satisfies the relation $DND' = N$, where N is not the normal form, is equal to $B^{-1}AB$, B being unique for each N.

4. Dr. H. R. Brahana: *Riemann surfaces and the map problem.*

Heawood solved the map problem for a surface of genus 1, and pointed out the method of procedure for $p > 1$. There is required a "map of verification" for the minimum number of colors for each value of the genus greater than 1. Heffter gave these maps for each value up to 6. This paper translates

the problem into one of selecting systems of branch points and cuts on a sphere so that the Riemann surface thus defined constitutes the required map on a surface of given genus. The solutions are given for genus equal to 1, 2, and 3. The paper also contains a definition of a Riemann surface based on analysis situs alone.

5. Mr. H. T. Davis: *An application of fractional differentiation to a class of Volterra integral equations.*

A theory of fractional differentiation is developed which removes a difficulty of Riemann's definition. The theory is then used to obtain solutions of Volterra integral equations of the second kind with infinite discontinuities in the kernel.

6. Professor G. A. Miller: *Same left co-set and right co-set multipliers for any given finite group.*

This paper will appear in full in an early issue of this BULLETIN.

7. Dr. M. M. Feldstein: *The invariants of the linear group modulo p^k.*

The n-ary linear homogeneous group modulo p^k, $G(n, p^k)$, is decomposed into factor-groups in two ways. The first decomposition brings to light the relations among the invariants of $G(n, p^k)$ and those of $G(n, p^{k-1})$. The second decomposition serves to derive the necessary and sufficient conditions which the invariants of $G(n, p^{k-1})$ must satisfy in order to be invariant under $G(n, p^k)$. Special treatment is required for $G(n, 2^k)$. A fundamental set of invariants is set down.

8. Professor S. Lefschetz: *Continuous transformations of manifolds.*

In this note there is given a new method for treating questions of continuous transformations of an n-dimensional manifold W_n. In particular it is shown how the determination of the minimum number of fixed points is reduced to a readily solved problem concerning intersections of cycles of a W_{2n}.

9. Mr. M. H. Ingraham: *Certain limitations of the value of the complete independence of a set of postulates.*

The notion of complete independence was introduced partly in order to do away with certain trick methods of securing ordinary independence. It eliminated the use of vacuous fulfilment of a postulate. Since then methods of questionable value, such as strengthening of postulates and introducing

irrelevant elements have arisen for use in securing complete independence. Examples of such methods in recent papers are shown as well as examples against which there can be no criticism. The completely independent set of postulates for positive integers which was previously presented by the writer have such undesirable features and another set is presented from which these features have been eliminated.

10. Professor L. E. Dickson: *Foundations and status of the theory of linear algebras.*

In the usual definition of a linear algebra A of order n over a field F, each element of A is an ordered set $(x_1, \cdots, x_n)$ of n numbers of F. In recent work of Scorza (PALERMO RENDICONTI, vol. 45, p. 7, and *Corpi Numerici e Algebre*, Messina, 1921) the elements u_i of A are undefined entities, combined by addition and multiplication, and by a scalar multiplication producing products of u_i and numbers α_i of F, certain associative and distributive laws being postulated for these operations. Another postulate is that A contains n elements u_i such that every element of A is expressible in one and only one way in the form $\sum \alpha_i u_i$. The present paper replaces this strong postulate by the much weaker one that A has a finite basis. Various theorems are proved for algebras A over any field F which have been proved hitherto only when F is the field of all complex numbers. The results of this paper and those of the following paper will appear in the author's book entitled *Algebras and Their Arithmetics*, to be published in July by the University of Chicago Press.

11. Professor L. E. Dickson: *General theory of hypercomplex integers.*

Books on the arithmetic of quaternions were published by R. Lipschitz in 1886 and by A. Hurwitz in 1919. In a long series of extensive memoirs, Du Pasquier attempted to extend Hurwitz's methods to various algebras other than quaternions, but without success since under his definition most aglebras do not have integers, while if integers do exist, they are usually without interest since the laws of arithmetic fail and cannot be restored by the introduction of ideals however defined. This criticism is explained in full in the author's memoir in the JOURNAL DE MATHÉMATIQUES, 1923, where there is presented a new wholly satisfactory theory of integers in any algebra over the field of complex numbers. The present paper extends this theory to algebras over any field.

12. Dr. Irwin Roman: *The optical transformation of surface differentials.*

The problem of geometrical optics is that of studying the new wave train when we know the incident wave train and the refracting surface. The usual law of refraction is contained in the invariance of the vector product of $r_i h_i$ and H_i where r_i is the refractive index of the ith medium, h_i is the unit normal vector of the wave incident on the ith surface, and H_i is the unit normal vector for the interface between the ith and $(i+1)$th media, at the point of incidence. The author combines a vector and a differential geometry method, obtaining the first and second differential coefficients of the refracting surface from the corresponding values for the interface and the incident wave, along with the distance function for this pair. For the special case of pure propagation, without refraction, the results furnish the usual formulas for parallel surfaces. The formulas simplify when the lines of curvature are taken as parametric curves, and also when the surface is one of revolution. A number of special cases are considered.

13. Professor F. R. Moulton: *Solutions of ordinary differential equations.*

If the right members of the differential equations are analytic in the dependent and independent variables, or if they have the Lipschitz property in the dependent variables and are continuous in the independent variable, then it is shown by a process depending on two parameters that the solution exists over the domain of definition of the differential equations. As a by-product the validity of the method of numerical solution of differential equations is established. It is shown by very direct means that the solutions and their first derivatives with respect to the independent variable are continuous functions of the independent variable and of the initial values of the variables, individually and conjointly. The extensions to the cases where the right members of the differential equations have continuous partial derivatives with respect to the dependent variables are included.

14. Professor Dunham Jackson: *On approximation by functions of given continuity.*

The problem of this paper, in its simplest form, is the approximation of an arbitrary continuous function $f(x)$ by means of functions satisfying a given Lipschitz condition. The discussion is concerned with the existence, uniqueness, and con-

vergence properties of the function of closest approximation according to the criterion of least mth powers. The treatment is susceptible of extension in various directions.

15. Mr. D. Kazarinoff: *Vector treatment of the extrema of double integrals.*

In this paper it is shown that a double integral of the type studied by G. Kobb (ACTA MATHEMATICA, vol. 16 (1892)) with the integrand satisfying his conditions (4), (ibid., p. 69), can be reduced to $\int\int_S F(x, y, z, X, Y, Z)d\sigma$ and vice versa. Here X, Y, Z are the direction cosines of a normal to S and F is such that, k being any real constant, $XF_X + YF_Y + ZF_Z = kF$. This theorem is a consequence of one given by J. Radon, MONATSHEFTE, vol. 22 (1911), pp. 53–63.

The author considers then the integral $\int\int_S F(r, n)d\sigma$ where F is a scalar function of r, a vector with components x, y, z, and n, the unit vector with components X, Y, Z, and $\text{grad}_n F \cdot n = kF$, and he derives a differential equation which a surface S must satisfy to furnish an extreme value for the integral. The author also considers some particular cases of this equation and its transformation into more explicit form, yielding a geometric interpretation.

16. Professor F. E. Wood: *Cubics associated with a net of conics.*

There is, in general, a unique cubic $K = 0$ associated with a net of conics, with the property that the first polars of the points of the plane with respect to $K = 0$ form the conics of the net. The equation of this cubic, the *fundamental cubic* of the net, and some of its properties, are derived.

If the point $P(\lambda_1, \lambda_2, \lambda_3)$ is associated with the conic $\lambda_1\varphi_1 + \lambda_2\varphi_2 + \lambda_3\varphi_3 = 0$ of a net of conics, then the points P which correspond to the degenerate conics of the net form a cubic curve in the λ-plane. This is called a Δ *cubic* of the net. In this paper the properties of the Δ cubics are obtained, and certain relations existing between the fundamental cubic, the Δ cubics and the Jacobian cubic of the net. The fundamental cubic of a net will, in general, uniquely determine and will be uniquely determined by a net of conics, which is true for the Jacobian cubic or the Δ cubics in special cases only.

17. Professor J. S. Turner: *An extension of the theory of the double modulus.*

In this paper, certain theorems of the classical double modulus theory are extended to the double modulus $m, P(v)$, where

m is any positive integer, and $P(v)$ is a rational and integral function of v, with integral coefficients, irreducible modulo m. Other theorems are extended to the double modulus p^2, $P(v)$, where p is a prime, and $P(v)$ is irreducible modulo p.

18. Professor Henry Blumberg: *On certain properties of sets of positive measure.*

Let $M_1, M_2, \cdots, M_n$ be n planar sets of positive measure (Lebesgue). It is then shown that there exist in the plane n circles $C_1, C_2, \cdots, C_n$ such that, if $P_1, P_2, \cdots, P_n$ are any n points lying respectively in these circles, there is a congruent set of points $Q_1, Q_2, \cdots, Q_n$ lying respectively in $M_1, M_2, \cdots, M_n$. Again, if M is a planar set of positive measure and $S = \{P_1, P_2, \cdots, P_n\}$, any finite set of points in the plane, there exists in M a set similar to S. Other properties are obtained. The results of the paper hold for n-space.

19. Professor I. A. Barnett: *On a class of invariant subgroups of the conformal and projective groups in function space.*

Kowalewski has defined the conformal group in function space as the totality of all regular infinitesimal transformations which leave invariant the angle between two curves in function space. In this paper all the subgroups of the conformal group are found which leave invariant the manifold in function space $\int_0^1 [f'(x)/k(x)]\,dx = 1$. The explicit forms of these subgroups are given for the cases $n = 1, 2$. It is shown, furthermore, that for $n > 2$, there are no subgroups having the required property. An analogous discussion is made for the projective group of function space.

20. Professor I. A. Barnett: *The area-preserving group in function space.*

This paper studies all the regular infinitesimal transformations which leave invariant the areas of triangles in function space. These are $\delta f(x) = [\alpha(x) + \int_0^1 \beta(x, y) f(y)dy]\delta t$, where $\alpha(x)$ and $\beta(x, y)$ are arbitrary continuous functions of their arguments such that $\beta(x, y) + \beta(y, x) = 0$. These transformations form a group in the sense defined by Kowalewski which is, in fact, the analog of the group of motions in n-space. The preceding transformations also leave invariant all the curvatures of a skew curve in function space.

ARNOLD DRESDEN,
Secretary of the Section.

THE APRIL MEETING OF THE SOCIETY IN NEW YORK

The two hundred twenty-ninth regular meeting of the Society was held at Columbia University, on Saturday, April 28, 1923, extending through the usual morning and afternoon sessions. The attendance included the following fifty-seven members:

Archibald, Beal, Birkhoff, Blichfeldt, B. H. Camp, Cole, Crum, Dantzig, Dodd, Douglas, Fine, Fiske, Fite, M. C. Foster, Philip Franklin, Fry, Gafafer, Gill, Gronwall, Hazlett, Hebbert, E. R. Hedrick, Hille, Huntington, Joffe, Kellogg, Kircher, Kuhn, Lamson, MacDuffee, Mathewson, Meder, Mirick, Molina, Mullins, Northcott, Oglesby, Osgood, Pell, Pfeiffer, Raynor, Reddick, R. G. D. Richardson, Ritt, Rosenbaum, Seely, Siceloff, Silverman, Sosnow, Swartzel, Tracey, Vandiver, H. E. Webb, Weisner, Wetzel, H. S. White, J. W. Young.

At the meeting of the Council, the following twenty persons were elected to membership in the Society:

Professor Thomas Cicero Amick, Elon College;
Professor Harry Cyrus Bradley, Massachusetts Institute of Technology;
Mr. Alexander Joseph Cook, Harvard University;
Professor Palmer Hampton Graham, New York University;
Professor David Arthur Hatch, Lafayette College;
Mr. Harold Hotelling, Princeton University;
Professor Emma Hyde, Kansas State Agricultural College;
Mr. Philip Chapin Jones, Goodyear Tire and Rubber Company;
Mr. Ralph Eugene Kennon, University of Iowa;
Mr. Edgar Lucien Larkin, Lowe Astronomical Observatory;
Mr. Ben Zion Linfield, Harvard University;
Mr. George Averett Lyle, Lehigh University;
Mr. Dalip Singh Saund, University of California;
Mr. David Skolnik, Central High School, Newark;
Mr. Han Yih Tsang, Columbia University;
Mr. John Isaac Vass, Northwestern University;
Professor Evelyn Walker, Hunter College;
Mr. Albert Harry Wheeler, North High School, Worcester;
Professor Hugh Brown Wilcox, University of Minnesota;
Mr. Kitizi Yanagihara, Yamagata Higher School.

The Secretary announced that the following members of the London Mathematical Society had entered the American Mathematical Society since the February meeting, under the reciprocity agreement:

Dr. Bevan Braithwaite Baker, University of Edinburgh;

Professor John Edward Aloysius Steggall, University College, Dundee; Professor William Henry Young, University College, Aberystwyth.

Twenty-seven applications for membership in the Society were received.

The Secretary reported the appointment by President Veblen of the following Committee on Endowment: Professors J. L. Coolidge (chairman), Arnold Dresden, and G. C. Evans, and Messrs. Robert Henderson and G. E. Roosevelt (treasurer). The Council approved a statement of uses to which income from the proposed endowment may be devoted. Briefly these include the subsidizing of the TRANSACTIONS, the Colloquium Lectures, and other research publications, together with treatises on advanced topics. A statement from Professor Coolidge outlining the proposed programme of the Committee was read.

Professor Edward Kasner was elected a member of the Editorial Committee of the TRANSACTIONS, as successor to Professor L. P. Eisenhart, who had declined renomination.

Professor Eisenhart was elected to complete the term of service of Professor C. N. Haskins as representative of the Society in the Division of Physical Sciences of the National Research Council, Professor Haskins having asked to be relieved because of ill health.

The Secretary reported that the Cole Prize Fund (see this BULLETIN, vol. 29, p. 14) had been augmented to more than one thousand dollars. The following committee was appointed to set the first prize problem and to arrange the conditions of award: Professors H. S. White (chairman), Blichfeldt, Dickson, Fiske, and Osgood.

About fifty members took luncheon together between the sessions, and fifteen gathered at the dinner after the meeting.

Professor H. S. White presided at the morning session, relieved in the afternoon by Professor E. V. Huntington. Titles and abstracts of the papers read at this meeting follow below. Mr. Michal was introduced by Professor Evans, Professor Nörlund by Professor Birkhoff, Dr. Rainich by Professor Kasner, and Mr. Levy by Professor Eisenhart.

Professor Haskins' paper was read by Professor Silverman, and the papers of Bliss, Hutchinson, Glenn, Gronwall (first paper), Douglas (second paper), Evans and Bray, Michal, Wiener, Walsh, Garabedian, Safford, Graustein, Nörlund, Eisenhart, Brinkmann, Lipka and Hollcroft, Zeldin, and Veblen and Thomas were read by title.

1. Professor W. L. Crum: *The resemblance between the ordinate of the periodogram and the correlation coefficient.*

This paper points out that the ordinate of the periodogram, as now defined, does not differ materially from the product-sum which occurs in the numerator of the correlation coefficient for the original data with a periodic function of the simplest trigonometric type. This fact indicates that a slight change in the definition of the periodogram may be desirable, and suggests also an ideal means of studying the lag in the correlation of two historical series.

2. Professor G. A. Bliss: *Birational transformations simplifying singularities of algebraic curves.*

There is a well known theorem in the theory of algebraic curves which states that every such curve can be transformed by a birational transformation into one which has no singularities except double points with distinct tangents. The theorem is not a simple one to demonstrate, and many of the proofs which have been given are incomplete or inaccurate. In a preceding paper * the author has commented upon these proofs and has signalized two of them as being especially interesting. One is by Walker, who developed an alteration, suggested by Klein in 1894, of a method originally devised by Bertini for the projective plane. In the second, by Hensel and Landsberg, reasoning proposed by Kronecker in 1881 is extended to apply to curves in the function-theoretic plane. Both of these proofs are lengthy and complicated when all the details are taken into consideration. In the present paper the author has remodeled the method of Kronecker so that it can be applied to both planes, and has attained what he hopes will be regarded as simpler proofs of the two corresponding theorems.

* *The reduction of singularities of plane curves by birational transformations*, this BULLETIN, vol. 29 (1923), pp. 161–183.

3. Professor O. D. Kellogg: *Curvature and the top.*

If a plane or spherical curve has curvature, K, which has a continuous derivative with respect to the arc, s, and if for $a \leqq s \leqq b$ this derivative is positive, the osculating circle at the point s_2 lies within the osculating circle at s_1 if $a \leqq s_1 < s_2 \leqq b$. The present paper proves this theorem, and gives applications to the theory of the top suggested by Professor Osgood's article in the April number of the TRANSACTIONS OF THIS SOCIETY (vol. 23, No. 3).

4. Professor E. V. Huntington: *Simplified proof of l'Hospital's theorem on indeterminate forms.*

L'Hospital's familiar theorem on indeterminate forms is usually based on the generalized law of the mean, the proof of which often appears to the student as a highly ingenious device. The present author's proof of this theorem depends only on the most elementary properties of integrals, and the motive for each step taken is comparatively obvious. The theorem is stated in an extended form so as to include curves with vertical points of inflection, and new examples are given to show the necessity of the condition that the derivative of the denominator must not vanish in the neighborhood of the point in question.

5. Professor E. V. Huntington: *Tables of Lagrangean coefficients, for interpolating without differences.*

The chief advantages of the Lagrangean formula for interpolation (for equal intervals) are (1) that the required interpolated value is expressed directly in terms of the tabulated values, without the necessity of forming a table of differences, and (2) that the formula, unlike those of Stirling, Bessel, etc., can be written down at once, without effort, and verified by inspection. The chief disadvantages are (1) the fact that the numerical factors required are not small numbers as in the case of the ordinary formulas, and (2) that tables of the coefficients have not hitherto been available. The first of these disadvantages has largely disappeared with the advent of computing machines, and the second is removed by the tables presented by the present author. By the aid of these tables an interpolated value can be computed on the machine in less time than it takes to write out the differences. The tables are arranged for polynomials of the third, fourth, and fifth degrees.

6. Professor J. I. Hutchinson: *A remarkable class of entire functions.*

If $f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$ has real and positive coefficients and if $a_n^{-1} = b_1 b_2 \cdots b_n$, it is proved that $b_n \geqq 4b_{n-1}$ $(n = 2, 3, \cdots)$ are necessary and sufficient conditions that $f(z)$, as well as any polynomial formed by any number of consecutive terms of $f(z)$, has all its roots real and distinct (except $z = 0$, when it is a multiple root). Various properties and special cases are considered.

7. Professor C. N. Haskins: *On Newton's formulas for the sums of the powers of the roots of an algebraic equation.*

The theorem of this note was brought out by an attempt to simplify and check Bourguet's * computation of the coefficients of the power-series development of the Γ function.

8. Professor O. E. Glenn: *Invariants of the transformation of a differential form by analytic transformations.* Preliminary report.

It is known that restrictions which reduce the number of operations of the group or set of transformations applied to a quantic enlarge the totality of independent concomitants. If conditions are placed upon the arbitrary functions in the transformations for a binary differential form, such as to make these functions satisfy the pair of conditions for being analytic, the differential parameters increase in numbers and new methods are required. The author develops theory for the identification of complete systems in certain domains.

9. Dr. T. H. Gronwall: *Mutual induction of two square coils.*

This paper gives a method for the rapid numerical computation of the mutual induction of two square coils with parallel axes, one coil being rotated through any angle about its axis. Such an arrangement is used in a certain type of radio antenna. The paper will appear in the BELL SYSTEM TECHNICAL JOURNAL.

10. Dr. Einar Hille: *On Dirichlet's series with complex exponents.*

The author considers series of the type $\sum a_n e^{-\lambda_n s}$, where the exponents λ_n form a sequence of complex numbers such that $|\lambda_n| \to \infty$. It is shown that the region of absolute convergence of such a series is a convex domain. This region

* ANNALES DE L'ECOLE NORMALE, (2), vol. 10 (1881), pp. 175–233.

is determined in terms of the a_n and the λ_n under different assumptions concerning the exponents. For every convex region Dirichlet's series can be constructed, having the region as region of absolute convergence. The results are extended to Dirichlet's integrals.

11. Dr. Jesse Douglas: *Normal congruences and quadruply infinite families of curves in space.* Second paper.

In a previous paper of the same title (this BULLETIN, vol. 28, p. 238) the author classified quadruply infinite families of space curves with respect to the normal congruences contained within them. In that classification, two types of curve family, termed natural and quasi-natural, were predominant. The present paper solves the following problem: To determine, for each of the following four properties, all families of ∞^4 curves in space which have that property. (1) The ∞^2 curves of the family which pass through an arbitrary point form a normal congruence. (2) The ∞^2 curves of the family which meet an arbitrary plane perpendicularly form a normal congruence. (3) The ∞^2 curves of the family which meet an arbitrary sphere perpendicularly form a normal congruence. (4) The ∞^2 curves of the family which meet an arbitrary straight line perpendicularly form a normal congruence. There are three types with the property (1), of which two are the natural and quasi-natural. Five types, including the natural and quasi-natural, have the property (2). All and only natural and quasi-natural families have the property (3). All and only natural families have the property (4).

12. Dr. Jesse Douglas: *Determination of all families of ∞^4 curves in space in which the sum of the angles of every triangle is two right angles.*

This paper will appear in full in an early issue of this BULLETIN.

13. Dr. C. C. MacDuffee: *On the complete independence of the functional equations of involution.*

The functional equations (1) $\varphi(a, m)\cdot\varphi(a, n) = \varphi(a, m + n)$, (2) $\varphi(\varphi(a, m), n) = \varphi(a, mn)$, (3) $\varphi(a, n)\cdot\varphi(b, n) = \varphi(ab, n)$ are satisfied unreservedly by the numbers of the class C_+ of positive numbers and zero, when $\varphi(a, n)$ is identified with the power function a^n. The complex numbers satisfy these equations when proper determinations of the multiple-valued power functions are used. In this paper, a non-linear algebra

N is introduced whose elements are the pairs of real numbers $[\xi, r]$. Two operations, addition and multiplication, are defined, and it is shown that these operations obey all the ordinary laws of algebra except the associative law of addition. This algebra N is then used in the discussion of the complete independence of equations (1), (2) and (3). With certain underlying conditions, it is proved that the functional equations are not completely independent, for (1) and (2) imply (3). The remaining seven cases are non-empty, for seven functions are defined on the algebra N which obey the underlying conditions, each fulfilling one of these seven cases.

14. Professor G. C. Evans and Dr. H. E. Bray: *The Dirichlet problem for the sphere, and its generalization; necessary and sufficient conditions.*

The authors find necessary and sufficient conditions that a function harmonic within a sphere, not necessarily bounded, may be written as a Poisson integral in terms of boundary values f summable on the surface of the sphere in the Lebesgue sense; also, more generally, as a Poisson integral in terms of the differential of an additive function of point sets $F(\omega)$ on the surface of the sphere. Extending the result of Fatou, they show that the harmonic function takes on the value of the point set derivative of $F(\omega)$ wherever the latter exists, and hence, in the former problem, the value of f almost everywhere. A physical interpretation in terms of sources and doublets is given for the extended problem, using a new theorem of Vitali on the analysis of continuous functions of limited variation with zero derivative almost everywhere.

15. Mr. A. Michal: *Integro-differential invariants of one-parameter groups of Volterra transformations.* Preliminary report.

The author commences the study of a problem allied to the calculus of variations, by showing that there always exist functionals of a function y and its derivative which are invariant under a given arbitrary continuous one-parameter group of linear functional transformations of Volterra type of the argument y. The functional is assumed to involve y and its derivative between the values 0 and x, and, considered as a functional of these two arguments, may for a given functional value of the derivative of y be assigned arbitrarily as a functional of the other argument.

16. Dr. Norbert Wiener: *A generalization of the Dirichlet problem.*

The author uses the methods of generalized integration due to Daniell to extend the notion of a harmonic function corresponding to given boundary conditions to discontinuous boundary conditions of a very general type. Existence theorems are given.

17. Dr. Norbert Wiener: *A new type of summability.*

The author develops a type of summability with trigonometric weighting factors, and shows that it correctly evaluates the Fourier series of a continuous function. This type of summability is not stronger than the first type of summability of Cesàro.

18. Dr. J. L. Walsh: *On the location of the roots of Lamé's polynomials.*

A certain problem in the statical equilibrium of particles arises naturally in connection with the theory of equations. This same problem, in a generalized form, arises also in connection with the location of the roots of certain polynomial solutions of a differential equation considered by Lamé. This paper extends to the latter type of problem some simple results recently established for the former type.

19. Mr. C. A. Garabedian: *Rods of constant or variable circular cross section.*

In a paper on *Circular plates of constant or variable thickness*, presented to the Society February 24, 1923, the author developed a method of series in elasticity, and called attention to the possibility of treating by the same method the problem of the thin rod. Cylindrical coordinates are adopted, and the rod is generated by revolution of a curve $r = f(z)$ about the axis of z. The chief interest lies in the fact that the section may vary with z. Writing $r = \rho\tau$, $f(z) = \varphi(z)\tau$, it is assumed that the displacements are developable in ascending powers of the parameter τ. The equations of equilibrium become identities on τ, and computation of the coefficients in the formulas of displacement is direct. In prescribing boundary conditions at the ends, conventional use is made of Saint-Venant's principle. The total differential equations which arise can be integrated explicitly in a wide range of cases. Applications are made to the uniformly tapering rod, to a bulging rod, and to the right circular cylinder.

20. Professor F. H. Safford: *A pendulum of varying length.*

This paper was written in answer to a request to obtain the law of variation in length if the amplitude of swing is to increase as rapidly as possible. In solving the differential equation of motion, it was found that a change of the independent variable time could be made which reduced the problem to that of a simple pendulum of fixed length. The amplitude may be made any assigned function of the time, and a criterion was obtained concerning the transition from vibration to complete revolution.

21. Professor W. C. Graustein: *Note on a certain type of ruled surface.*

This paper will appear in full in an early issue of this BULLETIN.

22. Professor N. E. Nörlund: *On certain difference equations.*

The object of this paper is to study the solutions of the difference equation $\Delta^n \omega F(x) = \varphi(x)$, where $\varphi(x)$ is a given function. By a certain method of summation the author defines an operation which is inverse to the difference operation Δ. He finds thus a particular solution of the above equation. This solution may also be defined by boundary value conditions. The paper contains a detailed discussion of the properties of this particular solution. The general solution differs from the above particular solution by an arbitrary function whose nth difference is zero.

23. Professor L. P. Eisenhart: *Orthogonal systems of hypersurfaces in a Riemann space.*

Let $g_{rs}dx^r dx^s$ be the fundamental quadratic form of a Riemann space of order n, and α_{rs} the covariant components of any symmetric tensor of the second order other than g_{rs}. The equations $(\alpha_{rs} + \rho g_{rs})\lambda^r = 0$ determine an n-uple of congruences whose directions at any point are mutually orthogonal, the functions $\lambda_h{}^r$ being the contravariant components of the tangents to a curve of the congruence C_h at any point. The necessary and sufficient conditions upon the tensors g_{rs} and α_{rs} in order that the congruences be normal, and consequently the space admit an n-uply orthogonal system of hypersurfaces, are determined in this paper. It is shown that the problem is an algebraic one. The conditions are simple when the roots of the equation $(\alpha_{rs} + \rho g_{rs}) = 0$ are simple or double at most, but are quite involved where there are roots of the third and higher orders.

24. Professor L. P. Eisenhart: *Symmetric tensors of the second order whose first covariant derivatives are zero.*

The following theorem is established: A necessary and sufficient condition that a Riemann space admit a symmetric covariant tensor of the second order α_{rs} other than the fundamental tensor of the space, g_{rs}, such that its first covariant derivative is zero, is that the fundamental quadratic form $g_{rs}dx^r dx^s$ be reducible to a sum of quadratic forms φ^i, the coefficients of each form being functions at most of the x's of that form; then $\alpha_{rs}dx^r dx^s = \sum_i \rho_i \varphi^i$, where the ρ's are arbitrary constants. The determination of whether or not a given space is of this kind is reducible to a problem of algebra.

25. Dr. G. Y. Rainich: *Geometry of curved space without coordinates.*

An n-dimensional bundle of vectors is introduced axiomatically, the properties of addition of vectors, multiplication of vectors with scalars, and formation of ratio serving as axioms. Space is considered as a totality of points; an axiom states that vectors issuing from one point constitute an n-dimensional bundle. This permits introducing a relation between different bundles. The transitivity of this relation is the condition that space be euclidean. In the general case the geometry in the vicinity of every point can be given by a trilinear vector function which is the generalized Riemann tensor. Scalar and vector multilinear functions are introduced as two kinds of tensors which are essentially distinct unless a length unit is introduced. Weyl's "Dichten" are shown on the contrary not to be essentially different from tensors. The notions of transformations, invariants, covariant and contravariant quantities are not essential to geometry and arise only with the introduction of coordinates. The metric does not define the relation between bundles, but for the representation of a metric we might use these relations, and in this case the Riemann tensor can be deduced from this representation.

26. Mr. Harry Levy: *Normal congruences of curves in a Riemann space.*

The following theorems are demonstrated: (1) A necessary and sufficient condition that a congruence of curves in a Riemann space of order n admit a family of r-dimensional $(r > 1)$ hypersurfaces as orthogonal trajectories is that

$\gamma_{ijk} = \gamma_{ikj}$ $(i = 1, 2, \cdots, n - r;\ j,\ k = n - r + 1, \cdots, n)$, where the functions γ_{ijk} have the significance given them by Ricci and Levi-Civita (MATHEMATISCHE ANNALEN, vol. 54 (1901), p. 148). (2) If each of n mutually orthogonal congruences has a family of r-dimensional hypersurfaces as orthogonal trajectories, and if the n families of hypersurfaces are distinct, the congruences are normal, that is, each has a family of $(n - 1)$-dimensional hypersurfaces as orthogonal trajectories.

27. Dr. Philip Franklin: *Linear tensor equations.*

In this paper we investigate under what conditions a set of equations involving tensor components, invariant, as a set, under changes of coordinates, is equivalent to a set of tensor equations, i.e., equations expressing the vanishing of *all* the components of a tensor. We show that a set of equations *linear* in the components of a single tensor, with numerical coefficients, holding for all coordinates, is equivalent to a set of tensor equations. This theorem, proved for n-space, is applied to the proof that there are essentially only three distinct linear tensor equations in the curvature tensor, a theorem proved for 4-space by Birkhoff. A general theorem restricting the possible forms of linear tensor equations is added, which may be used to deduce such equations as those of the electromagnetic field.

28. Dr. Philip Franklin: *A qualitative definition of the potential functions.*

The purpose of this paper is to set up assumptions characterizing the potential functions which do not involve derivatives or integrals, and express obvious properties of the physical quantities giving rise to potential functions. It is shown that a class of functions of two variables such that its members are continuous, any linear combination of two members is in the class, any member gives rise to a new member if the coordinate axes are shifted (orthogonal transformation of the variables), it contains a constant function, and for an infinite sequence of functions of the class, taking values on the boundary of a fixed circle approaching zero uniformly over the boundary, the values at the center of the circle cannot approach a limit different from zero, *is necessarily a set of harmonic functions.* An analogous set of assumptions is given for potential functions of three variables.

29. Mr. H. W. Brinkmann: *Riemann spaces conformal to Einstein spaces.*

A Riemann n-space which satisfies those Einstein equations often referred to as the "cosmological equations" is called an *Einstein* space. The author obtains a necessary and sufficient condition that a given Riemann space be conformal to such a manifold. This is then applied to the special case where the given manifold is an Einstein space to start with. For the dimensionality $n = 4$ there is proved, among other things, the following theorem: If two four-dimensional Einstein spaces are mapped conformally they are either both of constant Riemann curvature (spherical spaces) or they are isometric and the conformal map in question is merely a change of scale. A note on this subject has been sent to the PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES.

30. Mr. H. W. Brinkmann: *Einstein spaces mapped conformally on each other.*

In this paper the author determines all n-dimensional Einstein spaces which can be mapped conformally on another Einstein space provided that map which consists in a mere change of scale is ignored. The results are quite simple, and agree for $n = 4$ with the theorem stated in the preceding paper. It must be noted that the two Einstein manifolds in question need not have the same scalar curvature, but if their scalar curvatures are both zero or both not zero it is shown that one of the two conformal spaces is isometric to that obtained from the other by a suitable change of scale. The isometric map is, however, not established by the conformal one. When one of the scalar curvatures vanishes and the other does not, it is evident that the two spaces can not be isometric even if a change of scale is allowed.

31. Dr. M. C. Foster: *Surfaces with orthogonal loci of the centers of geodesic curvature of an orthogonal system.*

The displacements of the centers of geodesic curvature are considered relative to the moving trihedral. The loci of these points are not surfaces but orthogonal curves of which at least one must be a straight line. The given orthogonal system must be the lines of curvature. Such surfaces are *surfaces of Joachimsthal* with circular lines of curvature in one system, and for which the fundamental quantities satisfy the relation $(\partial/\partial u)((\partial r/\partial u)/\xi) + \xi r_1 = 0$.

32. Professor Joseph Lipka: *Trajectory surfaces.*

In a space of any dimensionality V_n, any two directions through a point P determine a pencil of directions, and the curves through P in the pencil of directions which minimize the integral $\int \phi ds$, where ϕ is any point function and ds is the element of length, determine a surface or spread of two dimensions, called a trajectory surface (a generalization of a geodesic surface). In this paper the author discusses the Gaussian curvatures of these surfaces and their relations to the Gaussian curvatures of the corresponding geodesic surfaces. It is found, e.g., that a necessary and sufficient condition that the curvatures of a trajectory surface and its corresponding geodesic surface should be the same is that the determining pencil of directions contain the direction of the vector ϕ. Furthermore, we have here a generalization of Ricci's median curvature, principal directions, principal congruences, and principal invariants in any space V_n; the theorems are analogous to those which Ricci has found in a discussion of geodesic surfaces.

33. Professor Joseph Lipka: *Geometric interpretation of the second differential parameter.*

Given any surface and any function ϕ of the coordinates on the surface, the invariant or second differential parameter of ϕ, usually designated by $\Delta_2\phi$, plays an important rôle in many problems in surface theory. The present paper contains a simple geometric interpretation of this parameter, which may be stated as follows: If any direction on the surface through a point P moves first by parallelism and then by conformal parallelism (whose characteristic function is ϕ) completely around an infinitesimal cycle drawn on the surface through P, the ratio of the angle formed by its final positions to the area of the cycle is the value at P of $\Delta_2\phi$.

34. Professor E. L. Dodd: *Formulas for the greatest and the least variate under general laws of frequency or error.*

The author classifies probability or relative frequency functions $\phi(x)$ as follows: For values of x sufficiently large, $\phi(x)$ is expressed as the product of an arbitrary $\psi(x)$, which satisfies certain inequalities, and (1) 0; (2) $x^{-1-\alpha}$; (3) g^{x^α}; (4) $g^{(\log_c x)^\gamma}$; (5) g^{c^x}; (6) x^{-x}; with $\alpha > 0$, $0 < g < 1$, $c > 1$, $\gamma > 1$; and then shows that for any positive η, there is for large enough n, a probability $> 1 - \eta$ that the greatest of n variates will be: (1) Constant, (2) $n^{1/\alpha}$; (3) $(-\log_g n)^{1/\alpha}$,

(4) c to the power $(-\log_g n)^{1/\gamma}$, (5) $\log_c(-\log_g n)$, (6) G in $G^G = n$, respectively; with a relative error small at pleasure for the values in (1), (3), (5), (6); and for $1/\alpha$ and $1/\gamma$ in (2) and (4). These results are applied to the seven Pearson types, the Bruns series (finite), the Jorgensen logarithmic function, the Poisson exponential (law of small numbers), the Charlier B-series (finite), and the Makeham life function. If G is determined from $\int_G^\infty \phi(t)dt = 1 - 2^{-1/n}$, it is, of course, equally probable that the greatest variate will or will not exceed G. This gives results for $ke^{-h^2x^2}$ in close agreement with those of Bortkiewicz.*

35. Professor T. R. Hollcroft: *Singularities that may be added to those of curves of given order.*

This paper will appear in an early issue of this BULLETIN.

36. Dr. S. D. Zeldin: *On the quadratic ternary partial differential equations admitting Lie groups of orders 4 and 5.*

The author determines the coefficients of the linear partial differential equation of the second order in three independent variables of the form $a_{11}z_{11} + a_{22}z_{22} + a_{33}z_{33} + 2a_{12}z_{12} + 2a_{13}z_{13} + 2a_{23}z_{23} + 2a_1z_1 + 2a_2z_2 + 2a_3z_3 + az = 0$ which is invariant under Lie groups of orders 4 and 5 of given structures.

37. Professor H. F. Blichfeldt: *On the approximate solution in integers of a set of m linear non-homogeneous equations in $n > m$ unknowns, and the final form of Kronecker's theorem.*

Kronecker's theorem asserts that if the coefficients of the equations in question satisfy certain "conditions of rationality," then a set of integers $x_1, x_2, \cdots, x_n$ exist which, when substituted for the unknowns, will satisfy the equations up to certain errors whose absolute values are all less than a previously assigned small positive quantity ϵ. In the present paper it is shown that if the errors are designated $\epsilon_1, \cdots, \epsilon_m$, and if $R = \sqrt{x_1^2 + \cdots + x_n^2}$, we may demand, in addition to $|\epsilon_i| < \epsilon$, also that $|\epsilon_1\epsilon_2 \cdots \epsilon_m| R < f$, a certain number depending upon the coefficients of the given equations, but independent of ϵ and of $x_1, \cdots, x_n$. When $m = 1$, f represents any preassigned positive number. It is furthermore shown that when $m > 1$, equations exist for which nothing may be demanded of the errors individually, beyond $|\epsilon_i| < \epsilon$; the case $m = 2$, $n > 3$ may furnish a possible exception.

* *Variationsbreite und mittlerer Fehler*, SITZUNGSBERICHTE DER BERLINER MATHEMATISCHE GESELLSCHAFT, vol. 21; October, 1921.

38. Professor Oswald Veblen and Mr. T. Y. Thomas *Geometries of paths admitting first integrals.*

In this paper, the condition for the existence of homogeneous linear and quadratic integrals of paths in the geometry of paths introduced by Professors Eisenhart and Veblen (PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), pp. 19–23) is reduced to the algebraic consistency of a set of equations, as is also the condition of existence of the general first integral subject to the particular restriction that its first covariant derivative vanish. Thus a necessary and sufficient condition for the geometry of paths to reduce to the Riemann geometry is given by the algebraic consistency of a set of equations. Other applications are also made. In addition the paper contains a discussion of normal coordinates for the geometry of paths from a different viewpoint from that adopted by Professor Veblen (PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), pp. 192–197) and a treatment of the subject of covariant differentiation in the geometry of paths in which a whole group of tensors derivable from a given tensor is obtained. The first tensor of this group is the ordinary covariant derivative. The paper will appear in the TRANSACTIONS OF THIS SOCIETY.

39. Dr. T. H. Gronwall: *Isothermal surfaces with spherical lines of curvature in one system.*

By the use of pentaspherical coordinates, it is shown that all these surfaces are obtainable by inversion from the isothermal surfaces with plane lines of curvature in one system, which were investigated by Darboux (*Théorie des Surfaces*, vol. 4, chapter X). The present method determines all such surfaces; it yields Darboux's results in a simpler manner, and also a new class of surfaces of this kind with simple geometrical properties.

40. Dr. T. H. Gronwall: *Extension of Tchebychef's statistical theorem.*

In the first part of this paper, inequalities stronger than that of Tchebychef are found under the assumption that all moments of even order $m_2, m_4, \cdots, m_{2k}$ are known, k being any integer. No assumption on the probability function is made (except that it is never negative). In the second part, a similar investigation is carried out assuming the probability function monotone decreasing beyond the limit of skewness.

R. G. D. RICHARDSON,
Secretary.

AN ELEMENTARY PROOF OF A FUNDAMENTAL LEMMA CONCERNING THE LIMIT OF A SUM*

BY H. J. ETTLINGER

It is the object of this note to give a simple proof of the following lemma upon which may be based the theory of the Riemann integral.

Let the interval $I: a \leqq x \leqq b$ *be divided into* n *equal sub-intervals,* $I(i, n)$, *each of length* $\Delta_n x = (b - a)/n$. *To each* $I(i, n)$ *there corresponds a number* $h(i, n)$. *Let* $|h(i, n)| \leqq M$ *for all values of* $i\ (\leqq n)$ *and* n, *where* M *is a constant. If* x *is any fixed value in* I, *then for each value of* n, x *is contained in at least one of the sub-intervals. Let it be designated by* $I(x, n)$. *For each fixed* x *and each subdivision* $I(x, n)$ *let the corresponding number be* $h(x, n)$, *and let* $\lim_{n\to\infty} h(x, n) = 0$. *Then*

$$\lim_{n\to\infty} \sum_{1}^{n} h(i, n)\Delta_n x = 0.$$

In a recent paper † the above lemma was stated in terms of area as a geometric theorem and used to develop the theorems on definite integrals of fundamental importance in the integral calculus. In addition, applications were given to the transformation of a double integral by change of variables and to the Fredholm method of solution of an integral equation.

A proof based on a generalization of a theorem due to W. H. Young ‡ has been given by R. L. Moore.§ For the proof of the above lemma, however, it is sufficient to use Arzelà's ‖ "lemma fondamentale" which may be stated in terms of the

* Presented to the Society, September 7, 1922.

† *A simple form of Duhamel's theorem and some new applications*, AMERICAN MATHEMATICAL MONTHLY, vol. 29 (1922), p. 239.

‡ *Open sets and the theory of content*, PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 2 (1904), pp. 20–22.

§ *On Duhamel's theorem*, ANNALS OF MATHEMATICS, (2), vol. 13 (1912), p. 163.

‖ *Un teorema intorno alle serie di funzioni*, LINCEI RENDICONTI, (4), vol. 1 (1885), pp. 262–267.

notation of the above theorem as follows:

If for each n, $I(n) = \sum_1^k I(m_i, n)$, where m_i is a positive integer such that

$$1 \leqq m_1 < m_2 < \cdots < m_k \leqq n,$$

and $I \geqq I(n) > \delta > 0$ for every n, there is at least one point P which is common to an infinite number of the $I(n)$'s.

The proof of the lemma may also be based on the following theorem of Osgood.*

If $s_n(x)$ is continuous in $a \leqq x \leqq b$ and converges when $n \to \infty$ for every x in the interval, and if $|s_n(x)| \leqq M$ for all values of $n = 1, 2, \cdots$ and x in $a \leqq x \leqq b$, where M is a constant, then

$$\lim_{n\to\infty} \int_a^b s_n(x)dx = \int_a^b \lim_{n\to\infty} s_n(x)dx.$$

This theorem is applied as follows. On $I(i, n)$ as base erect an isosceles triangle whose altitude is $2h(i, n)$. The equal legs of these triangles form the graph of $s_n(x)$. Identify

$$\sum_1^h h(i, n)\Delta_n x = \int_a^b s_n(x)dx.$$

Finally note that $\lim_{n\to\infty} s_n(x) = 0$ for every fixed value of x, and the lemma is proved.

Other proofs of Osgood's theorem have been given by F. Riesz,† Bieberbach,‡ and Landau.§ The method given by Landau is substantially that given by Moore (loc. cit.) and makes use of Arzelà's "lemma fondamentale" (loc. cit.). The proof by Bieberbach is essentially the same as that of Landau, with the exception that a simplified proof of Arzelà's theorem is included. The method of Riesz is more elementary and has suggested the proof given in this note.

It is to be noticed *mutatis mutandis* that Osgood's theorem

* *On non-uniform convergence and integration of a series term by term*, AMERICAN JOURNAL, vol. 19 (1897), p. 188.

† *Ueber Integration unendlicher Folgen*, JAHRESBERICHT DER VEREINIGUNG, vol. 26 (1917–18), pp. 274–278.

‡ *Ueber einen Osgoodschen Satz aus der Integralrechnung*, MATHEMATISCHE ZEITSCHRIFT, vol. 2 (1918), pp. 156–157.

§ *Ein Satz ueber Riemannsche Integrale*, MATHEMATISCHE ZEITSCHRIFT, vol. 2 (1918), pp. 350–351.

may be regarded as a corollary of the lemma of this note. It would seem to be more natural to establish the lemma stated above by elementary methods and derive from it the properties of the Riemann integral, of which the theorem of Osgood would be one.

We proceed to the proof. Let

$$f(n) = \sum_1^n |h(i, n)| \Delta_n x.$$

Then

$$f(n) \geqq |\sum_1^n h(i, n) \Delta_n x| \geqq 0.$$

If $\lim_{n \to \infty} f(n) = 0$ is proved, the lemma is established.

CASE 1. Let $|h(x, n+1)| \leqq |h(x, n)|$ for every fixed x in I. Then $f(n) \geqq f(n+1) \geqq 0$, or $f(n)$ is a monotonically non-increasing function of n. Hence

$$\lim_{n \to \infty} f(n) = A \geqq 0.$$

Divide I into two equal parts. Then $|h(i_2, 2)| \Delta_2 x \geqq A/2$, where i_2 has at least one of the two values, $i = 1, 2$. Divide $I(i_2, 2)$ into two equal parts. Then $|h(i_4, 4)| \Delta_4 x \geqq A/2^2$, where i_4 has at least one value of the set 1, 2, 3, 4. If this process be continued k times, we have $|h(i_m, m)| \Delta_m x \geqq A/m$, where $m = 2^k$ and i_m has at least one value of the set 1, 2, $\cdots, 2^k$. But $\Delta_m x = (b-a)/m$. Hence $A \leqq |h(i_m, m)| (b-a)$.

The closed sub-intervals $I(i_2, 2)$, $I(i_4, 4)$, $\cdots$, $I(i_m, m)$ are so related that each is contained in the preceding, and, as n increases without limit, $\Delta_m x$ approaches zero. Therefore there exists a point $\overline{P}$ of I whose abscissa is $\bar{x}$ such that $\overline{P}$ is common to all these sub-intervals. Hence $h(i_2, 2)$, $h(i_4, 4)$, $\cdots$ form a set of numbers $h(\bar{x}, n)$ corresponding to $\bar{x}$. But $\lim_{n \to \infty} |h(\bar{x}, n)| = 0$. Hence $A = 0$. This same method with a slight modification can be used for this case if the length of the subdivisions are unequal.

CASE 2. For the general case, let $h(x, n, m)$ for $n \leqq m$ be the greatest of the numbers $|h(x, n)|$, $|h(x, n+1)|$, $\cdots$, $|h(x, m)|$. Form

$$\phi(n, m) = \sum_1^n h(i, n, m) \Delta_n x,$$

where $h(i, n, m)\Delta_n x$ is the area of the rectangles whose bases fill up $I(i, n)$ and whose heights $h(i, n, m)$ are chosen so as to satisfy the preceding condition for each x in $I(i, n)$. Now $h(x, n, m) \leqq M$ for all values of m and $n \leqq m$. Hence

$$\phi(n, m) \leqq M(b - a).$$

For fixed x and n, $h(x, n, m) \leqq h(x, n, m + 1)$. Hence

$$\phi(n, m) \leqq \phi(n, m + 1).$$

So that for a fixed n, $\phi(n, m)$ is a monotonically non-decreasing function of m. Hence $\lim_{m\to\infty} \phi(n, m) = H(n) \leqq M(b - a)$, and

$$\phi(n, m) \leqq H(n). \tag{1}$$

But $f(n) \leqq \phi(n, m)$, or $f(n) \leqq H(n)$. Hence $\lim_{n\to\infty} H(n) = 0$ will carry with it $\lim_{n\to\infty} f(n) = 0$.

Given $\delta > 0$ and arbitrarily small, define a sequence of positive integers, m_i, to satisfy the inequality

$$\phi(i, m_i) \geqq H(i) - \frac{\delta}{2^i}, \qquad (i = 1, 2, \cdots, n). \tag{2}$$

Since $\phi(i, m) \geqq H(i) - (\delta/2^i)$ for $m > m_i$ we may choose the set m_i to satisfy $m_1 < m_2 < m_3 < \cdots < m_n < \cdots$. Let $\bar{h}(x, n)$ denote the smallest of the numbers $h(x, 1, m_1)$, $h(x, 2, m_2)$, $\cdots$, $h(x, n, m_n)$. Form

$$\psi(n) = \sum_1^n \bar{h}(i, n)\Delta_n x,$$

where, as above, $\bar{h}(i, n)\Delta_n x$ is the area of the rectangles whose bases fill up $I(i, n)$ and whose heights $\bar{h}(i, n)$ are chosen so as to satisfy the preceding condition for each x in $I(i, n)$.

We shall prove by induction that

$$\psi(n) > H(n) - \delta\left(1 - \frac{1}{2^n}\right). \tag{3}$$

For $n = 1$ we have $\bar{h}(x, 1) \equiv h(x, 1, m)$, and hence $\phi(1, m_1) \equiv \psi(1)$. Hence by (2), (3) is true for $n = 1$. If (3) is not true for every positive integer, let k be the first integer for which it is not true. Then (3) is true for $n = k - 1$, or

$$\psi(k - 1) > H(k - 1) - \delta\left(1 - \frac{1}{2^{k-1}}\right). \tag{4}$$

Now

$$\bar{h}(x, k) \leqq \bar{h}(x, k - 1), \quad \text{and} \quad \bar{h}(x, k) \leqq h(x, k, m_k). \tag{5}$$

But

(6) $\bar{h}(x, k-1) \leqq h(x, k-1, m_{k-1}) \leqq h(x, k-1, m_k)$.

Compare

$$\bar{h}(x,\ k-1) - \bar{h}(x,\ k) \geqq 0$$

and

$$h(x,\ k-1,\ m_k) - h(x, k, m_k) \geqq 0.$$

If $\bar{h}(x, k-1) \leqq h(x, k, m_k)$, then $\bar{h}(x, k-1) - \bar{h}(x, k) = 0$, and hence in this case

$$\bar{h}(x, k-1) - \bar{h}(x, k) \leqq h(x, k-1, m_k) - h(x, k, m_k).$$

If $\bar{h}(x, k-1) > h(x, k, m_k)$, then

(7) $$\bar{h}(x, k) = h(x, k, m_k).$$

Subtracting (7) from (6), we have

$$\bar{h}(x, k-1) - \bar{h}(x, k) \leqq h(x, k-1, m_k) - h(x, k, m_k).$$

Hence in every case

(8) $\bar{h}(x, k) \geqq \bar{h}(x, k-1) + h(x, k, m_k) - h(x, k-1, m_k)$,

for every x in I. Then

(9) $\psi(k) \geqq \psi(k-1) + \phi(k, m_k) - \phi(k-1, m_k)$.

By (4), (2), and (1) we may strengthen (9):

$$\psi(k) \geqq H(k-1) - \delta\left(1 - \frac{1}{2^{k-1}}\right) + H(k) - \frac{\delta}{2k} - H(k-1),$$

or

$$\psi(k) \geqq H(k) - \delta\left(1 - \frac{1}{2k}\right).$$

This proves (3). Now strengthen (3) and write

(10) $$\psi(n) > H(n) - \delta.$$

But

(11) $$\bar{h}(x, n) \leqq h(x, n, m_n).$$

Also

(12) $$h(x, n, m) = |h(x, \nu(x))|$$

where $n \leqq \nu(x) \leqq m_n$ and $\lim_{n\to\infty} \nu(x) = \infty$.

Now $\bar{h}(x, n)$ for every fixed x in I is a monotonically non-increasing function by (5), and $\lim_{n\to\infty} \bar{h}(x, n) = 0$ by (11) and (12). Hence we have, by Case 1, $\lim_{n\to\infty} \psi(n) = 0$ and $\lim_{n\to\infty} H(n) = 0$, by (10).

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KLEIN'S COLLECTED PAPERS, VOLUME II.

Felix Klein: Gesammelte mathematische Abhandlungen. Vol. II: *Anschauliche Geometrie; Substitutionsgruppen und Gleichungstheorie; Zur mathematischen Physik.* Edited by R. Fricke and H. Vermeil. Berlin, Julius Springer, 1922. vi + 713 pages; 185 figures.

Although one of the editors has been replaced in the second volume, the purpose has not been altered, and it appears under the same plan as the preceding one. The memoirs in the first part, mentioned under the first subtitle, are sixteen in number, being memoirs XXXIV to XLIX inclusive. Of these, nine were written during the years 1873–6, five in the interval 1892–5, and two since 1900. This first part occupies 250 pages.

The first memoir is a description of four models of the Plücker complex surface, constructed while the author was a student and assistant of Plücker in Bonn. The first is the general Kummer surface with sixteen nodes, which appears as the surface of singularities of a general quadratic line complex. The second is the surface enveloped by lines of the complex which meet a given line. The line is double on the surface, which has eight double points. The third is defined in the same way, when the given line is an ordinary line of the complex, and the fourth has for double line a singular line of the complex. These models were rather crudely constructed and very crudely reproduced in metal casts, but their importance in the study of singularities of algebraic surfaces is very great. About two pages have been added to the original text, which was prepared for the *Katalog mathematischer Modelle* . . ., von W. Dyck (1892).

The next paper, of 52 pages, is on cubic surfaces. To the original paper that appeared in volume 6 of the MATHEMATISCHE ANNALEN, eighteen pages have been added, which explain more in detail the surfaces with biplanar points, and discuss the configuration of the 27 real lines on the diagonal surface. These two papers are the only ones on the construction of models, but they are sufficient to show the intimate relation between spatial intuition and mathematical conclusion as they existed in the mind of the author.

The procedure is nearly always the same, starting from a singular or indeed composite surface and seeing what changes necessarily follow when one or more parameters change slightly. The method was sharply attacked when first employed, then reluctantly tolerated on account of its unquestioned results, until now it is accepted as a completely rigorous and trustworthy process, brought to a wonderfully high degree of perfection by Italian geometers. The Rodenberg collection of models of cubic surfaces, proposed by Clebsch, was made more complete and systematic by the use of the above memoirs.

When Klein was called to Erlangen, in his opening address he proposed the establishment of a general collection of models and apparatus to simplify the teaching of mathematics. Later the same idea was developed

much further in Munich, in collaboration with Professor A. Brill, whose brother L. Brill undertook the commercial manufacture of the models. While in Munich, Klein had for his assistant W. Dyck, who was an important factor in having models and apparatus recognized as an important and necessary part of mathematical instruction. The Brill factory in Darmstadt, later taken over by M. Schilling in Halle, under the scientific direction of Professor F. Schilling has supplied a large proportion of all the plaster models in use throughout the world. Since the death of Martin Schilling, and particularly since the war, the manufacture of a considerable number of the more complicated models has been discontinued. If this state of affairs is not corrected, it will mean a terrible loss to later generations of mathematical teachers.

The Dyck catalogue and a considerable collection of models was sent to Chicago as a part of the German exhibit at the World's Fair of 1893. Klein was the representative of the Prussian government, and was authorized to explain these models. Out of the talks and conferences occasioned by them grew the Evanston Colloquium.

The next two papers, on the connectivity of surfaces, and on relations among the singularities of algebraic curves, follow minutely the same ideas as those developed in the memoir on cubic surfaces. The most important element is spatial intuition. Now follow four papers, together over eighty pages, on the combination of analysis and spatial intuition, that are entirely convincing, are strikingly beautiful, and seem curiously simple when approached from this point of view. They are on a new form of Riemann surfaces, and their application to the study of Abelian integrals of genus 3.

The method is to study a composite quartic curve, consisting of two conics, and fix the rational parameters of the points of contact of the common tangents. Then replace the conics by a non singular quartic. Now bitangents appear, and points of inflexion, the arrangements of which can be partly determined immediately from the figure. The rational parameters are replaced by Abelian integrals, and the various cases are so clearly separated that one wonders why the subject can be so hard.

Much later (1892) the same ideas were applied to the normal curve of genus p. While the results here obtained are both striking and important, it soon becomes evident that the last word on the uniformization of algebraic curves has not yet been said.

A short paper on the discriminant variety of an algebraic curve, also prepared for the Dyck catalogue, and a note on the geometric interpretation of the successive convergents of a continued fraction complete the list of contributions to the study of space relations. The next forty pages are devoted to the theory of knowledge and to the foundations of mathematics. The point of view is that we learn by experience and perfect our knowledge by successive approximations. In his 1901–02 course on the applications of mathematics Klein advocated a theory of errors for graphical processes analogous to that of least squares for numerical data obtained from observation. Astronomers require extreme numerical nicety based on crude analytic formulas, yet the sequel has shown that they were most

frequently right. The same phenomenon is observed in connection with the singularities of plane curves. Both the original formulas of Plücker and the further relations established by Klein were first established empirically and later justified by analysis without the use of figures.

The first paper concerns the concept of a function, the result being that, so far as graphical or numerical control is concerned, we must deal with a band rather than with a mathematical curve. The substance of this paper, presented in 1873, was further developed in the Evanston Colloquium, and still further in the later paper on the arithmetization of mathematics. In the attempt to have this point of view crystalized the Beneke prize was offered in 1898, but was not awarded, as no memoir of required merit was submitted. This was partly due to the vagueness in the statement of the prize problem. The desired direction is indicated in the next paper, but the most significant statement in it is the old slogan "there is no royal road in mathematics." The last paper is a short address on the border questions of mathematics and philosophy delivered in Vienna in 1906. It emphasizes the difference between intuition and demonstration.

The second part, finite groups of linear substitutions, occupies 250 pages, and includes twelve memoirs, and some dozen pages of comments. The first of these memoirs was written in 1871, and the last in 1905. The first incentive to study the problems here considered is found in the short note written by Klein and Lie on those curves (*W*-curves) having the property that every tangent meets the sides of the triangle of reference in points which with the point of contact have a constant cross ratio. The thought is that curves invariant under finite groups of linear transformations have special properties, the study of which will contribute both to our knowledge of the curves and of the groups to which they belong. The use of projective geometry and of the geometry of inversion is fully justified by the results these subjects produced in the *Ikosaeder* and later *Modulfunktionen*. In particular, the linear transformations of the complex plane and their interpretation in terms of rotations of the regular polyhedra form the key note of the earlier papers. The important papers of Schwarz, making use of the 120 alternately congruent and symmetric triangles on the surface of the sphere, defined by the icosahedron, and later the classic on the algebraic solutions of the hypergeometric differential equation, were eagerly studied and compared with his own interpretations. The study received a big impetus when Gordan came to Erlangen in 1874; for several years he was one of the most eager workers in this field. Although Klein left Erlangen for the technical school of Munich in 1875, intimate association with Gordan was continued until 1880, when Klein moved to Leipzig.

During these years Brioschi and Kronecker were occupied with the quintic equation. On account of the similarity of the form of results in the two inquiries, it was suggested to Klein that knowledge of the solvability of the quintic equation would be of use to him in the study of the icosahedron, but he proposed the converse problem, namely, to make the icosahedron the basis of the study of the quintic equation. By recalling

the works of these men now it is fairly evident that each received great assistance from all the others, and there is no question that the geometric interpretation was an important factor for all of them. The most important illustration probably was the discovery of the simple ternary linear group of order 168, in connection with the transformation of order seven of the elliptic function, and the quartic curve which belongs to the group. How incomparably more instructive is the study of this curve than to confine one's attention to the general quartic. Klein refers to this period as the happiest and most fruitful of his life. But his intense activity began to undermine his health, and when he assumed new duties in Leipzig a break had to come. After a prolonged leave of absence, Klein realized that he would be obliged to work under lower pressure, and contented himself with systematizing and arranging results already obtained, rather than continuing the former program. In this period the *Ikosaeder* was written. Much of its substance had already appeared in memoirs, but in condensed form, and accompanied by considerable discussion of invariants. In the book the presentation is much more elementary, and most of the considerations of the theory of invariants is omitted. Klein protests against the idea that his book is only a visualization of a theory worked out by others; he maintains rather that it contains indispensable elements of the theory itself. At any rate Gordan succeeded in simplifying materially his own presentation after the book appeared. But Gordan was also able to simplify Klein's work on the sextic (MATHEMATISCHE ANNALEN, vol. 68 (1909)). This fact is graciously mentioned by Klein, who adds that the simplest and most elegant presentation is that given by Coble (MATHEMATISCHE ANNALEN, vol. 70 (1911)).

The study of the equation of the sixth degree was given to pupils, in particular to Reichardt * and to Cole.† Klein had made various attempts to employ quaternary linear groups in the study of the sextic equation, and dramatically describes the general surprise when Wiman called attention (MATHEMATISCHE ANNALEN, vol. 47 (1896)) to the existence of a simple linear ternary group of order 360, which had been found by Valentiner in 1889. Since this was first published in Danish, it was apparently entirely unknown until mentioned by Wiman, who showed that it was simply isomorphic with the even permutations of six letters. That the general solution of the sextic could be made to depend upon this group rather than upon a quaternary group was later proven by Klein in a letter to Castelnuovo. This letter was presented to the Accademia dei Lincei by Castelnuovo, and published in its RENDICONTI (ser. 5, vol. 8 (1899)). It appears in the present volume as memoir LX, and occupies page 480.

The last part of the volume, pages 505–713, is devoted to mathematical physics and contains memoirs LXII to LXXX inclusive. There are two

* *Ein Beitrag zur Theorie der Gleichungen sechsten Grades*, LEIPZIGER BERICHTE (1885), and *Ueber die Normierung der Borchardtschen Moduln der hyperelliptischen Funktionen vom Geschlecht* $p = 2$, MATHEMATISCHE ANNALEN, vol. 28 (1886).

† *A contribution to the theory of the general equation of the sixth degree*, AMERICAN JOURNAL, vol. 8 (1886).

sub-headings, one being Linear Differential Equations and the other General Mechanics. Klein had repeatedly stated that when he was a student his main interest was in physics, and that he deliberately planned an activity in mathematics for some years, as part of his preparation for work as a physicist. Although this plan could not be realized to any great extent, still, when we rehearse Klein's influence in furthering the knowledge and application of mathematical physics, we must concede that it was no mean accomplishment.

We are told in the introduction that the first stimulus was the association with Neumann while at Leipzig—1880 to 1886. The first papers were those on Lamé functions, closely followed by the monograph on Abelian integrals, all containing many physical concepts.

During these years Klein made several trips to France and to England, and he emphasizes how his ideas of mechanics were greatly clarified and extended by them, particularly through Maxwell and Hamilton. He takes pride in having had three important English books translated into German, Routh's *Dynamics*, Lamé's *Hydrodynamics* and Love's *Elasticity*. Two other books, written at a later time, were direct outgrowths of his courses, and have been standard books in their fields ever since. These are Pockel's *Ueber die Differentialgleichung* $\Delta u + k^2 u = 0$ *und deren Auftreten in der mathematischen Physik* (1891), and Bôcher's *Ueber die Reihenentwicklungen der Potentialtheorie* (1894). After Schwarz was called to Berlin (1892), Klein commenced a comprehensive plan of enlarging and broadening the mathematical instruction at Göttingen. Weber was called in 1892, succeeded by Hilbert in 1895, who largely directed the work in pure mathematics, while Klein devoted his energy to filling in the gap between mathematics and physics, assisted by Brendel, Sommerfeld, Riecke, Runge, and others. At this time Klein was sent to the Chicago exposition, and returned with American ideas of financing his new institute. Through the assistance of Althoff, minister of public instruction, he received liberal appropriations from the government, and through the assistance of Althoff as a private citizen he also gained the interest and generous financial support of several men prominent in the industrial world. By these two means he built up an extensive and symmetric institute, comprising a large number of branches of mathematics, and holding from its origin the front rank in the mathematical world. The encyclopedia was started in 1894; Klein suggested the plan, and personally assumed the editorship of the part on mechanics, which is just now nearing completion.

During the next decade he was active in so many organizations, and participated in so many international undertakings that his own productivity as an investigator decidedly diminished, yet as representative of the universities in the Prussian senate, as chairman of the International Commission on the Teaching of Mathematics, as member of the commission on the international catalogue of the Royal Society, and finally as chairman of the union of German teachers of science he exercised a tremendous influence in shaping mathematical instruction. His constant aim was not merely passively to understand the principles, but so to apply them as to make for continued progress, to intensify and enlarge life. Although these

activities were all in the interests of applied mathematics, yet Klein has continued the directorship and editorship of the MATHEMATISCHE ANNALEN since the death of Clebsch; he completed fifty years in this office last November.

The memoirs on Lamé functions are followed by those on the zeros of the hypergeometric series, the representation of the hypergeometric function by means of definite integrals, and the auto-reviews of the autographed lectures on the hypergeometric function and the linear differential equation of the second order, given in 1893–4. These were all published in the MATHEMATISCHE ANNALEN. The remaining essays were all published elsewhere. They include: a short report on recent English investigations on mechanics, a discussion of space collineations which occur in optical instruments, the greeting given at the opening of the mathematical congress at Chicago, the Princeton sesquicentennial lectures, two papers on graphical statics, one on Painlevé's criticism of Coulomb's law of friction, and finally one on the formation of vortices in frictionless liquids. The list is followed by a detailed explanation of the causes which led to the respective studies. The third and final volume is now in press. It contains the memoirs in the theory of functions.

VIRGIL SNYDER

LÉVY ON FUNCTIONALS

Leçons d'Analyse Fonctionnelle. By Paul Lévy, avec une préface de J. Hadamard. Paris, Gauthier-Villars, 1922. vi + 442 pp.

The increasing importance which is being given to the theory of functionals, or functions of lines, is illustrated by the fact that three of the Borel monographs in the last ten years have been concerned with this branch of mathematics, and the great breadth of the subject is illustrated by the fact that there is so little overlapping between the most recent of these, which is the subject of this review, and the earlier ones by Volterra,* and the more recent Cambridge Colloquium Lectures by Evans. In his introductory chapter, Lévy makes an interesting distinction between "algèbre fonctionnelle" and "analyse fonctionnelle." The first includes problems in which the unknowns are ordinary functions, but where the methods of the theory of functionals are used in determining them. The second includes problems where the unknowns themselves are functions of lines, or where the problems themselves could not be considered independently of the notion of a functional. Most of the work of Volterra and Evans mentioned above would belong to the "algèbre." The present monograph is primarily concerned with the "analyse."

The idea of a continuous functional is of fundamental importance. A functional $U(x(t))$ is said to be continuous if $U(y_n(t))$ approaches $U(x(t))$

* *Leçons sur les Equations Intégrales et les Équations Intégro-differentielles*, reviewed by Westlund in this BULLETIN, vol. 20 (1914), pp. 259–62, and *Leçons sur les Fonctions des Lignes*, reviewed by Bliss, ibid., vol. 21 (1915), pp. 345–55.

as a limit when the "distance" between $y_n(t)$ and $x(t)$ approaches zero. Thus there are as many definitions of a continuous functional as there are definitions of the "distance" between two functions. He mentions several different definitions which have been used, and develops the theory based on two of them to some extent. The definition which he prefers is that the distance r between $x(t)$ and $y(t)$ is given by the formula

$$r^2 = \int_0^1 [y(t) - x(t)]^2 dt.$$

In other words $U(x(t))$ is continuous if $U(y_n(t))$ approaches $U(x(t))$ when $y_n(t)$ approaches $x(t)$ in the mean. Under this definition continuous functionals of the first degree, that is, linear functionals, can always be expressed as Lebesgue integrals, homogeneous functionals of the second degree as double integrals, etc. According to the other definition the distance is the least upper bound of $|y(t) - x(t)|$, and consequently $U(y_n(t))$ does not need to approach $U(x(t))$ unless the sequence $y_1(t)$, $y_2(t)$, $\cdots$ approaches $x(t)$ uniformly, in order to be continuous. This makes it necessary to introduce Stieltjes integrals instead of those of Lebesgue. The author discusses the consequences of this definition under the heading, "le point de vue logique," explaining clearly Hadamard's theorem that a linear functional is the limit of a sequence of definite integrals, F. Riesz' proof that it is also expressible as a Stieltjes integral, and some of Fréchet's work on bilinear functionals. He does not consider this point of view so useful in the applications as "le point de vue pratique," and in the last part of Part I and in the other two parts of the book, he uses the definition based on approach in the mean. The appropriateness of these designations of the two points of view may be questioned, particularly by those who believe that the Stieltjes integral, which is now being used to good advantage by a few applied mathematicians, is destined to be of much greater use in applied science. At the present time, however, the "point de vue logique" appeals particularly to those who are interested in generalizing a mathematical theory for its own sake.

In general Part I is written in a very interesting as well as elementary manner, and is a very valuable introduction to this branch of mathematics.

The second part is mostly the result of the author's own researches, although he refers frequently to the work of Gateaux. The first type of equations considered in Part II is a generalization of equations involving total differentials. The equation

$$du(x_1, x_2, \cdots, x_n) = \Sigma p_i(x_1, x_2, \cdots, x_n, u)dx_i$$

has for its analog

$$\delta U[x(t)] = \int_0^1 f[x(t);\ U, t]\delta x(t)dt,$$

and its conditions of integrability are similar. A system such as

$$\frac{\partial u(x_1, \cdots, x_n, y_1, \cdots, y_n)}{\partial x_i} = f_i\left(x_1, \cdots, x_n, y_1, \cdots, y_n, \frac{\partial u}{\partial y_1}, \cdots, \frac{\partial u}{\partial y_n}, u\right) \quad (i = 1, 2, \cdots, n)$$

suggests the equation

$$U_x'(x(t), y(t); \tau) = f(x(t), y(t), U_y'; U, \tau),$$

where U_x' is the functional derivative of U considered as a function of $x(t)$, with $y(t)$ entering as a parameter, and U_y' defined similarly. The notions of characteristics, and complete integral, and Cauchy's method are extended to such equations.

If $x(t)$ belongs to a very general class of functionals defined on (0, 1), it can be approached in the mean by a sequence of functions $x_1(t), x_2(t), \cdots$ such that $x_n(t)$ is constant in each of the intervals

$$\left(\frac{r-1}{n} < x \leqq \frac{i}{n}\right).$$

Consequently if $U(x(t))$ is continuous, $U(x_n(t)) \to U(x(t))$ as $n \to \infty$. If then x_i is the value of $x_n(t)$ in the ith of these intervals, $U(u_n(t))$ may be considered as a function of n variables and may be called $u_n(x_1, x_2, \cdots, x_n)$. Since

$$\int_0^1 x_n^2(t)dt = \frac{1}{n}\sum_{i=1}^{n} X_i^2,$$

every point inside the sphere in "function space" whose equation is

$$\int_0^1 x^2(t)dt = R^2 \tag{1}$$

can be said to be approached by a sequence of points in the respective n-dimensional spheres

$$\sum_{i=1}^{n} x_i^2 = nR^2, \tag{2}$$

and the $U(x(t))$ will be approached by a sequence of functions $u_n(x_1, \cdots, x_n)$. Also the mean value of $U(x(t))$ in the sphere (1) may be defined as the limit of the mean value of $u_n(x_1, \cdots, x_n)$ in the spheres (2).

Part III begins with a discussion of the sphere in n dimensions, whose radius is $R\sqrt{n}$. If two such spheres have radii $R(1-\alpha)$ and R, where α is an arbitrarily small positive number, the ratio of their volumes will be $(1-\alpha)^n$ which approaches zero as $n \to \infty$. Thus it may be stated that all the volume of a sphere in function space is located arbitrarily near its surface, excepting a part of relative measure zero. Similarly it is proved that the same volume is concentrated arbitrarily near the equator. It can be proved by means of Stirling's formula that the ratio of the volume of an n-dimensional sphere to the expression $(2\pi e)^{n/2}R^n/\sqrt{n\pi}$ approaches unity as $n \to \infty$. It follows that the volume approaches zero as $n \to \infty$ if $R \leqq 1/\sqrt{2\pi e}$ and otherwise becomes infinite. Thus the volume of a sphere in function space may either vanish or be infinite. This does not interfere with the determination of the mean of a functional defined over the sphere, however, as the mean of the corresponding function in n-space may approach a definite limit. The mean is used in generalizing Green's formula, and the equation of Laplace in function space is discussed, and various applications are made.

The preface by Hadamard emphasizes the importance of the subject, and the value of the author's contributions to it.

C. A. FISCHER

SHORTER NOTICES

L'Oeuvre scientifique de Laplace. By H. Andoyer. Paris, Payot & Cie., 1922. 162 pp.

After a brief sketch of the life of Laplace, Andoyer sets forth the characteristics of the works of this great French scientist. Andoyer cites the problems in celestial mechanics, which the eighteenth century mathematicians encountered, and reminds the reader how, through insufficient approximation, doubt was cast for a time upon the validity of Newton's law of inverse squares, and how a closer numerical approximation dispelled those doubts. Andoyer presents evidence showing the excessive harshness of the judgment passed upon Laplace by certain writers, to the effect that Laplace, in his writings, often failed to give due credit to his predecessors and contemporaries. Laplace's relations to D'Alembert, Biot and Poisson are described. Andoyer explains how Laplace again and again returned to certain topics in order that he might improve his exposition and perhaps free the subject from metaphysical entanglements. Not altogether surprising is Laplace's lack of interest in certain abstract fields of mathematics, like the theory of numbers. But strange is Laplace's adherence to Newton's corpuscular theory of light a quarter of a century after Thomas Young had advocated the undulatory theory and a decennium after Fresnel had won Arago over to the latter theory.

Andoyer's masterly account of Laplace's researches on celestial mechanics, on the figure of the earth, on the tides, on the système du monde, on the analytical theory of probability, and of researches on physics contains numerous quotations from the works of Laplace, bearing on points of scientific and philosophical interest. Andoyer's booklet will be enjoyed by students interested in the evolution of the mathematical sciences. An alphabetical index would have enhanced still further the usefulness of the book.

FLORIAN CAJORI

La Composition des Mathématiques dans l'Examen d'Admission à l'Ecole Polytechnique de 1901 à 1921. By F. Michel and M. Potron. Paris, Gauthier-Villars, 1922. 12 + 452 pp.

This volume contains the questions asked in the examinations for admission to the Ecole Polytechnique for the twenty years from 1901 to 1921, together with complete solutions. It should be of great value anywhere as a source book of interesting and rather difficult problems in the fields of analytic geometry, calculus and dynamics. It will be chiefly illuminating to American readers, however, as exhibiting the very high standard in force in mathematical instruction in France. Probably only a very small percentage of our college graduates who have specialized in mathematics would be able to make a creditable showing on any of these examinations. The book therefore presents to us in this country a high ideal towards which we may aspire, even though under present conditions it seems far from attainable.

J. W. YOUNG

Elements of Projective Geometry. By G. H. Ling, George Wentworth, and D. E. Smith. Boston, Ginn and Company, 1922. vi + 186 pp.

Although there is no date on the title page, nor in the preface, this book, according to the notice of the copyright by Ginn and Company, was apparently published in 1922. The omission of the date of publication, however, does not matter, since the contents and treatment are such that the *Elements* might have been written in Steiner's time. Indeed the topics and their presentation are in the main those of the early period of synthetic geometry, which is well known to critical students of projective geometry.

Within its elementary restricted domain the book is written clearly and concisely and may serve very well as a first introduction to a modern university course in projective geometry. The reviewer is nevertheless of the opinion that such an introductory and elementary treatise could be written with a view to its bearing upon the foundations of geometry and on the channels which lead to the fountainheads of contemporary geometric science. Even in an elementary course an occasional and opportune outlook into higher domains is invigorating and creates in the ambitious student a desire to penetrate deeper into the subject.

Welcome features of the book under review are the large number of exercises and a short chapter on the history of projective geometry. Clifford's term *cross ratio* seems to us preferable to Chasles' *anharmonic ratio*. Harmonic is a more or less mystic designation for $(ABCD) = -1$, and anharmonic is a negative definition for an infinite number of cross ratios which are not equal to -1. In the short historic sketch the important fact ought to be pointed out that von Staudt made the first successful attempt to establish the foundations of projective geometry (Geometrie der Lage) by introducing the concept of "Wurf" = throw of four collinear points independent of metric concepts (cross ratio of metric segments). After the remarkable developments by Poncelet, Steiner, Moebius, Chasles, etc., von Staudt's geometry of position was by far the most important step in the early critical development of projective geometry.

ARNOLD EMCH

Elementary Vector Analysis. By C. E. Weatherburn. London, G. Bell and Sons, 1921. xxvi + 184 pp.

The author's object in this book is to present the simpler portions of vector analysis and to apply them to portions of mechanics. He adheres to the notation of Gibbs. He gets as far as differentials and integrals, but does not bring in the notions of curl, convergence, and other ideas that belong to the general study of fields. The definitions are geometric for the scalar and the vector products, the vectors being always thought of as lines, or geometric vectors. A summary at the end of nineteen pages makes it very easy to find formulas and definitions. The geometry of curves in space is treated briefly, kinematics and dynamics of a particle, systems of particles, rigid kinematics, rigid dynamics, rigid statics.

The book should serve as a simple introduction to these subjects treated by way of the vector methods, and for the purposes in view is admirably adapted to the student's needs.

JAMES BYRNIE SHAW

Die elliptischen Funktionen und ihre Anwendungen. Zweiter Teil. By Robert Fricke. Leipzig, B. G. Teubner, 1922. viii + 546 pp.

Almost exactly 171 years ago Euler was led to the discovery of the addition theorem for elliptic integrals of the first kind (*Institutiones Calculi Integralis*) by the researches of Count Fagnano. Hence Jacobi designated the 23d of December, 1751, as the birthday of elliptic functions. During this long period of development the most fruitful subject has been the theory of the transformations of elliptic functions. Enriched and organized by related fields, in particular by the theory of groups, the subject of elliptic functions as it appears today forms one of the most beautiful edifices ever erected by the human intellect.

The second volume of Fricke's treatise is devoted to the addition theorem and to the transformation theory. It has an introductory chapter of 155 pages containing concepts, definitions, and theorems from the theory of groups, algebraic functions, algebraic numbers, etc. The first part (131 pp.) deals with the addition theorem and the related multiplication and division theorems. The remainder of the book is devoted to the transformation theory.

The first volume appeared in October, 1915, and was reviewed for this BULLETIN (vol. 23 (1917), p. 319). The method of presentation and the underlying thought of the three volumes was there stated in some detail.

Volume II appeared in 1922, and maintains the characteristic Klein-Fricke style. Much use is made of geometry in picturing the behavior of groups by means of their fundamental domains. The book is clearly printed on excellent paper, and there are very few errors.

L. WAYLAND DOWLING

Fluoreszenz und Phosphoreszenz im Lichte der neueren Atomtheorie. By Peter Pringsheim. Berlin, Julius Springer, 1921. 202 pp. 32 figs.

Although a few interesting analogies to the phenomena of fluorescence and phosphorescence exist in the theory of small vibrations, classical dynamical principles, applied to either the elastic solid or the electromagnetic theory of light, have never been successful in accounting for the observed effects. The reason for this is that these phenomena are dependent upon the emission and absorption of light, and, so far, only the newer quantum dynamics appears to offer the solution.

A comprehensive account of the important results in this field of research is contained in this small book by Pringsheim, together with indications of their interpretation with the aid of the quantum hypothesis. According to this, the effect of the incident light is to displace the electrons in the atoms from their normal orbits; on falling back into these the fluorescent light is emitted in accordance with Bohr's principle. No account is given of the related phenomena of thermo-luminescence, nor luminescence produced by other than light radiation.

Such a resumé of the newer work upon a fairly wide field of research is of very great value, particularly when done as well as this. A useful feature is a list of 266 articles bearing upon this subject that have appeared since 1908. For earlier work, reference should be made to the fourth volume of Kayser's *Handbuch der Spektroskopie.*

E. P. ADAMS

NOTES

The third number of volume 23 of the TRANSACTIONS OF THIS SOCIETY (April, 1922) contains the following papers: *Some generalizations of geodesics*, by E. J. Wilczynski; *On the gyroscope*, by W. F. Osgood; *The relative distribution of the real roots of a system of polynomials*, by C. F. Gummer; *A general theory of conjugate nets*, by E. P. Lane; *Parallel maps of surfaces*, by W. C. Graustein.

The Alfred Ackermann-Teubner prize of the University of Leipzig has been awarded to Professor Paul Koebe, for his three memoirs *Über die Uniformisierung der algebraischen Kurven*, published in volumes 67, 69, and 72 of the MATHEMATISCHE ANNALEN.

The Lasserre prize has been awarded by the minister of instruction of France to Professor Paul Langevin, of the Collège de France.

At Cambridge University, the Adams prize for this year has been awarded to Professor J. Proudman, of the University of Liverpool; the subject was *Tidal theory*. The subject announced for this prize for the period 1923–1924 is *The physical state of matter at high temperatures*. Smith's prizes have been awarded to Mr. J. C. Burkill, Trinity College, for an essay on *Functions of intervals and the problem of area*, and to Mr. A. E. Ingham, Trinity College, for an essay on *Mean value theorems in the theory of the Riemann zeta-function*. Rayleigh prizes have been awarded to Mr. E. F. Collingwood, Trinity College, for an essay on *The formal factorization of an integral function of finite integral order*, to Mr. W. R. Dean, Trinity College, for an essay on *The elastic stability of a plane plate*, to Mr. E. C. Francis, Peterhouse, for an essay on *The Denjoy-Stieltjes integral*, to Mr. C. G. F. James, Trinity College, for an essay on *The analytical representation of systems of space curves*, and to Mr. M. H. A. Newman, St. John's College, for an essay *On discontinuities of functions of a single real variable*.

Professor William Cain, of the University of North Carolina, has been awarded the J. James R. Croes medal of the American Society of Civil Engineers for his paper on *The circular arch under normal loads*.

Professor H. Ludendorf, of the Astrophysical Observatory at Potsdam has been elected member of the Berlin Academy of Sciences.

Professor Albert Einstein delivered a course of lectures at Madrid in March. On this occasion he was elected a member of the Madrid Academy of Science.

Professor Florian Cajori, of the University of California, has been elected Vice-president of Section L of the American Association for the Advancement of Science.

The following 19 doctorates with mathematics as major subject were conferred by American universities in the academic year 1921–1922; there is added in each case the month and year in which the degree was granted, the minor subject (or subjects) if outside of mathematics, and the title of

the dissertation. C. R. Adams, Harvard, June 1922, *The general theory of the linear partial q-difference equation and of the linear partial difference equation of the intermediate type.* B. H. Brown, Harvard, June 1922, *The equilong transformations of Euclidean space.* Margaret Buchanan, Bryn Mawr, June 1922, Physics, *Systems of two linear integral equations with two parameters and symmetrisable kernels.* E. H. Carus, Chicago, December 1921, *Invariants as products: a vector interpretation of the symbolic method.* W. E. Cederberg, Wisconsin, June 1922, Astronomy, *On the solution of the differential equations of motion of a double pendulum.* H. S. Everett, Chicago, June 1922, *Determination of all general homogeneous polynomials expressible as determinants whose elements are homogeneous polynomials.* V. D. Gokhale, Chicago, March 1922, *Concerning compact Kürschák fields.* Claribel Kendall, Chicago, September 1921, *Certain congruences determined by a given surface.* R. E. Langer, Harvard, June 1922, I. *Developments associated with a boundary problem not linear in the parameter.* II. *The boundary problems and developments associated with a system of linear differential equations of the first order.* C. C. MacDuffee, Chicago, September 1921, *Invariantive characteristics of linear algebras with the associative law not assumed.* E. D. Meacham, Chicago, June 1922, *Properties of surfaces whose osculating ruled surfaces belong to linear complexes.* Eugenie M. Morenus, Columbia, June 1922, *Geometric properties completely characterizing the set of all the curves of constant pressure in a field of force.* Anna M. Mullikin, Pennsylvania, June 1922, *Certain theorems relating to plane connected point sets.* Eleanor Pairman, Radcliffe, June 1922, *Expansion theorems for solution of a Fredholm's linear homogeneous integral equation of the second kind with kernel of special non-symmetric type.* H. P. Pettit, Illinois, June 1922, Physics and Statistics, *A general cyclide with special reference to the quintic cyclide.* J. F. Reilly, Iowa, July 1921, Physics, *On certain generalizations of osculatory interpolation.* W. P. Udinski, Illinois, June 1922, Chemistry and Electrical Engineering, *On a series of rational functions formally analogous to Fourier's series.* B. C. Wong, California, May 1922, Astronomy, *A study and classification of ruled quartic surfaces by means of a point-to-line transformation.* E. B. Zeisler, Chicago, June 1922, Astronomy, *Definite integral representation of invariants.*

The following five doctorates, conferred in the latter half of the calendar year 1922, are listed in the same way as the preceding. E. H. Clarke, Chicago, September 1922, Astronomy, *On the minimum of the sum of a definite integral and a function of a point.* J. D. Eshleman, Chicago, September 1922, *The Lagrange problem in parametric form in the calculus of variations.* W. S. Kimball, Chicago, September 1922, Physics, *Scattering of particles by an Einstein curve.* Harry Langman, Columbia, November 1922, *Conformal transformations of period n and groups generated by them.* J. S. Turner, Chicago, *Fundamental system of formal invariants of a modular group of transformations.*

The following four doctorates in mathematical physics or mathematical astronomy were conferred during the calendar year 1922; there is added in each case the month in which the degree was conferred, the major sub-

ject, the minor subject, and the title of the dissertation. F. E. Carr, Chicago, December, mathematical astronomy, pure mathematics, *A solution of the problem of two bodies one of which is a rotating oblate spheroid.* H. C. Levinson, Chicago, September, mathematical astronomy, pure mathematics, *The gravitational field of masses relatively at rest according to Einstein's theory of gravitation.* Louis Slichter, Wisconsin, June, mathematical physics, applied mathematics, *An experimental study of an acoustic system.* Warren Weaver, Wisconsin, June, mathematical physics, applied mathematics, *A summary of the analytic formulation of the theory of electrodynamics.*

Professor K. Heun, of the Karlsruhe Technical School, has retired from active teaching.

Dr. P. Finsler has been admitted as privat docent at the University of Cologne.

Cambridge University has invited Professor H. A. Lorentz, of the University of Haarlem, to deliver the Rede Lecture, on May 15; the subject will be *Maxwell's electromagnetic theory.* Mr. A. Hutchinson, of Pembroke College, has been appointed university lecturer in crystallography.

At New Hampshire State College, Mr. W. E. Wilbur, Mr. Hubert Huntley, and Mr. P. F. Howard have been appointed instructors in mathematics. Professor H. I. Slobin, head of the department of mathematics, has been reappointed Director of the Summer School for the 1923 session.

Assistant Professor G. W. Mullins, of Barnard College, Columbia University, has been promoted to an associate professorship of mathematics.

Mr. W. F. Shields has been appointed professor of freshman mathematics at Fordham University. He succeeds Rev. L. J. McGarry, who has gone to Woodstock College, Maryland.

Assistant Professor F. N. Bryant, of Syracuse University, has been promoted to a full professorship of mathematics.

Dr. L. L. Smail, assistant professor of mathematics at the University of Washington, has been appointed assistant professor of mathematics at the University of Oregon, effective October 1, 1923.

Professor Max Abraham, formerly professor of mechanics at the Milan Technical School, died November 16, 1922, at Munich, at the age of forty-seven years.

Dr. Fritz Cohn, professor of theoretical astronomy at the University of Berlin, and director of the Berlin Rechen-Institut, died December 14, 1922, at the age of fifty-seven years.

Dr. James Gow, formerly headmaster of Westminster School, and author of *A short History of Greek Mathematics*, died February 16, 1923.

Mr. Rawdon Levett, one of the founders of the Association for the Improvement of Geometric Teaching, later the (British) Mathematical Association, and its first secretary, died February 1, 1923.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ASSIER DE POMPIGNAN (—.). Note sur le calcul tensoriel. Paris, Hermann, 1923. 8vo. 32 pp.

BOUCHARD (J.). Tables centésimales pour le tracé des courbes. Paris, Gauthier-Villars, 1922. 8vo. 204 pp.

BOUTROUX (P.). Les mathématiques. Paris, Albin Michel, 1922. 16mo. 184 pp.

BRYAN (G. H.). Mathematical tables. London, Macmillan, 1922. 28 pp.

EDWARDS (J.). A treatise on the integral calculus. Volume 2. London, Macmillan, 1922.

FERMAT (P.). Oeuvres de Fermat, publiées par les soins de Paul Tannery et Charles Henry: Documents inédits, publiés avec notices sur les nouveaux manuscrits par C. de Waard. Paris, Gauthier-Villars, 1922. 4to. 216 pp.

HENRY (A.). Calculus and probability for actuarial students. Published by the authority and on behalf of the Institute of Actuaries. London, C. and E. Layton, 1922. 7 + 152 pp.

HENRY (C.). See FERMAT (P.).

KRICHEWSKY (S.). A method of curve fitting. Cairo, Government Press, 1922. 4to. 17 pp.

LASLEY (J. W.). Some special cases of the flecnode transformation of ruled surfaces. (Diss., Chicago.) Chicago, 1922. 19 pp.

LORING (F. H.). Definition of equivalence. London, H. O. Lloyd, 1922. 16 pp.

ROMANE (R.). Géométrie du sens commun. Mulhouse, Brinkmann, 1922.

ROSNY (J. H.). Les sciences et le pluralisme. Paris, Alcan, 1922. 4 + 219 pp.

STUYVAERT (M.). Les nombres positifs. Manuel d'arithmétique. 3e édition. Gand, Van Rysselberghe et Rombaut, 1923. 8vo. 185 pp.

TANNERY (P.). See FERMAT (P.).

THOMPSON (S. P.). Calculus made easy. 2d edition, enlarged. London, Macmillan, 1921. 8vo. 301 pp.

VALENTINER (S.). Vektoranalysis. 3te, umgearbeitete Auflage. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 132 pp.

DE WAARD (C.). See FERMAT (P.).

WINGER (R. M.). An introduction to projective geometry. Boston, Heath, 1923. 14 + 443 pp.

WOODHOUSE (T.). Main counts and calculations. (Oxford Technical Manuals.) London, H. Frowde, and Hodder and Stoughton, 1921. 8 + 119 pp.

ZENON. Approximation dans l'interpolation. Paris, Gauthier-Villars, 1922. 8vo. 10 pp.

PART II. APPLIED MATHEMATICS

ABRAHAM (M.). Theorie der Elektrizität. Band 1: Einführung in die Maxwellsche Theorie der Elektrizität, von A. Föppl. 6te, umgearbeitete Auflage. Leipzig, Teubner, 1921. 8 + 389 pp.

ANDOYER (H.). Cours d'astronomie. 1re partie: Astronomie théorique. 3e édition, entièrement refondue. Paris, Hermann, 1923. 8vo. 455 pp.

ANDRADE (J.). Les organes réglants des chronomètres. Besançon et Bienne, E. Magron, 1922. 12mo. 146 pp.

D'AQUINO (L.). Il fondamento, il contenuto, le consequenze della nuova teoria della relatività. Esposizione critica elementare. Napoli, Detken e Rocholl, 1922. 8vo. 18 pp.

BATSON (R. G.) and HYDE (J. H.). Mechanical testing. Volume 1: Testing of materials of construction. London, Chapman and Hall, 1922. 13 + 413 pp.

BAXTER (H. E.). See YOUNG (G.).

BEHREND (B. A.). The induction motor and other alternating current motors. 2d edition. New York, McGraw-Hill, 1921. 23 + 272 pp.

BELLENOT (H.). See SOMMERFELD (A.).

BENRATH (A.). Physikalische Chemie. Heidelberg, Ehrig, 1922. 88 pp.

BIGOURDAN (G.). Mélanges scientifiques. Paris, Gauthier-Villars, 1922.

BRISSON (B.). See MONGE (G.).

CARTAN (E.). Sur les équations de la gravitation d'Einstein. Paris, Gauthier-Villars, 1922. 4to. 60 pp

CRAWFORD (W. J.). Elementary graphic statics. 2d edition. London, C. Griffin, 1923. 8vo. 140 pp.

DANVILLIER (A.). See LEDOUX-LEBARD (R.).

DE DONDER (T.). Premiers compléments de la gravifique einsteinienne. Paris, Gauthier-Villars, 1922. 4to. 38 pp.

DRUMAUX (P.). L'évidence de la théorie d'Einstein. Paris, Hermann, 1923. 8vo. 72 pp.

DUPORT (H.). Critique des théories einsteiniennes. La relativité et la loi d'attraction universelle. Dijon, Imprimerie Darantière, 1923. 8vo. 40 pp.

FABER (—.). Reinforced concrete simply explained. London, H. Frowde, and Hodder and Stoughton, 1922. 77 pp.

DA FANO (G.). Aufgaben aus der darstellenden Geometrie. 2te Auflage. Leipzig und Zürich, Speidel und Wurzel, 1921. 66 pp.

FÖPPL (A.). See ABRAHAM (M.).

GUYE (C. E.). L'évolution physico-chimique. La relativité d'Einstein dans la classification des sciences. L'évolution chimique et les probabilités. Le principe de Carnot envisagé dans les organismes vivants. Paris, Chiron, 1922. 116 pp.

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LAFON (C.). Etude sur le ballon captif et les aéronefs marins. Paris, Gauthier-Villars, 1922. 8vo. 208 pp.

LEDOUX-LEBARD (R.) ET DANVILLIER (A.). La physique des rayons X. Paris, Gauthier-Villars, 1921. 7 + 441 pp.

LORING (F. H.). Definition of relativity. London, H. O. Lloyd, 1922. 16 pp.

MAILLARD (L.). Quand la lumière fut . . . Tome I: Les cosmogonies anciennes. Paris, Les Presses Universitaires de France, 1922. 8vo. 214 pp.

MARGOULIS (W.). Les helicoptères. Recherches expérimentales sur le fonctionnement le plus général des hélices. Etudes sur la mécanique de l'hélicoptère. Paris, Gauthier-Villars, 1922. 11 + 90 pp.

MONGE (G.). Géométrie descriptive. Augmentée d'une théorie des ombres et de la perspective extraite des papiers de l'auteur par Barnabé Brisson. I–II. (Les Maîtres de la Pensée Scientifique.) Paris, Gauthier-Villars, 1922. 16 + 144 + 138 pp.

MÜLLER (A.). Die philosophischen Probleme der Einsteinschen Relativitätstheorie. Braunschweig, Vieweg, 1922. 8 + 224 pp.

NICEFORO (A.). Scopi e applicazioni del metodo statistico. Palermo, 1922.

PENDER (H.). Direct-current machinery. A text-book on the theory and performance of generators and motors. New York, Wiley, 1923. 10 + 314 pp.

RÉVEILLE (J.). Dynamique des solides. Gyroscopes. Paris, Baillière, 1923. 8vo. 506 pp.

SOMMERFELD (A.). La constitution de l'atome et les raies spectrales. Traduit sur la 3e édition allemande par H. Bellenot. Fascicule I. Paris, A. Blanchard, 1923. 8vo. 384 pp.

SPURGEON (E. F.). Life contingencies. Published by the authority and on behalf of the Institute of Actuaries. London, C. and E. Layton, 1922. 27 + 477 pp.

STEINMETZ (C. P.). Four lectures on relativity and space. New York, McGraw-Hill, 1923. 130 pp.

STEWART (A. W.). Some physico-chemical themes. London, Longmans, 1922. 12 + 419 pp.

THOMPSON (S. P.). Electrical tables and memoranda. 2d edition. London, Spon, 1923. 8 + 136 pp.

WILLIS (E. J.). The mathematics of navigation. Richmond, Va., Fergusson, 1921. 8vo. 34 pp.

WINTERNITZ (—.). Relativitätstheorie und Erkenntnislehre. Leipzig, Teubner, 1922.

YOUNG (G.) and BAXTER (H. E.). Descriptive geometry. New York, Macmillan, 1921. 309 pp.

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By

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